The math must be right: Comment on "Dimensional analysis, falling bodies, and the fine art of not solving differential equations" by C. F. Bohren [Am. J. Phys. 72, 534–537 (2004)]

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Abstract. we find it necessary to advise the interested and active instructor of Physics on the wrongness of some computations in the aforementioned article. Surprisingly, the Journal refuses to even publish an erratum on the paper, which naturally brings to mind the question of the number of published papers requiring corrections.

Keywords: Physics Education Research; Students Performance; Mathematics and Physics.

In discussing the usefulness of dimensional analysis for performing computations in physics, Bohren ended his interesting and inspiring article pointing out that "Unfortunately, because of the sheer drudgery of solving equations physical interpretation often is an afterthought instead of occupying pride of place, as it does in dimensional analysis."[1] In reply to this point of view and favoring that in the teaching of physics courses, instructors do need to reinforce the idea that developing a taste for solving equations is not only an integral part but also a very important aspect in the understanding of physics, one could cite a recent letter where the author regrets having lost the chance of wining the Nobel Prize because of following John Wheeler's advice to "never calculate without first knowing the answer."[2]

In the spirit of not following Wheeler's advice, we redid some of Bohren's computations in section II of his article. Starting from his equation (10), one can perform a Taylor's expansion on the right hand side of the expression $\tau\sqrt{2g/R} = cosh^{-1}(1/(1-2(h/R))$ to obtain,

$$\tau = \sqrt{\frac{2h}{g}}\left(1 + \frac{5}{6}(\frac{h}{R}) + \frac{43}{40}(\frac{h}{R})^2 + \frac{177}{112}(\frac{h}{R})^3 + \cdots\right). \tag{1}$$

This result contains the right numerical factor for the leading error term $\epsilon = \frac{5}{6}(\frac{h}{R})$, as computed by equation (2) in the article. Instructors might find it instructive to compare the result of equation (1) with the one leading to Bohren's wrong result expressed in his equation (11). It is obtained by performing a Taylor's expansion on both side of his equation (10).

To find out about the correctness of the result given by our equation (1) one can resort to the exact solution of this problem. Before doing that, let's mention that the posed problem (i.e. the falling of a point particle in the gravitational field of a uniform mass distribution spherically symmetric) is a completely solvable Newtonian mechanics problem, which solution only requires the knowledge of some integrals and the chain rule for derivatives[7, 3]. Nevertheless, this problem is missing from the list of illustrative examples in practically most commonly used textbooks. The exact solution can be written in the form,

$$\tau = \sqrt{\frac{L^3}{2GM}} \left[\frac{\sqrt{\frac{L}{x} - 1}}{\frac{L}{x}} + tan^{-1} \left(\sqrt{\frac{L}{x} - 1} \right) \right], \tag{2}$$

where both L, the position at which the point particle is released from rest at $\tau = 0$, and x, the position of the point particle at any later time τ, are measured from the center of the uniform mass distribution M having spherical symmetry of radius R, and G is the gravitational constant.

Now, by taking $L = R + h$ and $x = R$, equation (2) can be written in the form,

$$\tau = \sqrt{\frac{R^3(1 + h/R)^3}{2GM}} \left[\frac{\sqrt{\frac{h}{R}}}{1 + \frac{h}{R}} + tan^{-1} \left(\sqrt{\frac{h}{R}} \right) \right]. \tag{3}$$

Taylor's expansion of this expression yields (here $g = GM/R^2$, is the local gravitational constant)

$$\tau = \sqrt{\frac{2h}{g}} \left(1 + \frac{5}{6}(\frac{h}{R}) - \frac{1}{40}(\frac{h}{R})^2 + \frac{9}{560}(\frac{h}{R})^3 + \cdots \right), \tag{4}$$

thus confirming the result reported via equation (1) to leading terms, and providing corrections for additional higher order terms.

In presenting the previous computations, the active instructor should not miss the chance of talking about the fact that numerical factors are important when deciding on competing theories explaining physical phenomena. A remarkable example is provided by two models competing to explain the drag C_D on spherical bodies falling in a resistive medium:[4, 5] the Oseen approximation, equation (5a), which was discarded in favor of the correct model provided by Proudman and Pearson, equation (5b),

$$C_D = \frac{6\pi}{R_e} \left(1 + \frac{3}{8}R_e - \frac{19}{320}R_e^2 + \frac{71}{2560}R_e^3 + \cdots \right), \tag{5a}$$

$$C_D = \frac{6\pi}{R_e} \left(1 + \frac{3}{8}R_e + \frac{9}{40}R_e^2 ln(R_e) + \cdots \right), \tag{5b}$$

where R_e represents the Reynolds number (introduced in section VI of Bohren's article). Certainly, experiments help in choosing the right model based on the numerical answer provided by each model as compared by the respective measurement. Additional examples illustrating this idea can be drawn from the tests that are applied to competing theories which try to explain the perihelion shift of the planet Mercury and the deflection of light by the sun. Einstein's General Relativity theory is the winner on the basis that its predictions are in agreement with the results of increasingly accurate experiments[6].

Let's finish this comment by pointing out that, aside from the above discussion, the examples presented in Bohren's article [1] could be applied to help students to practice what they have learned in their math courses. Let's recall that it is in physics courses where students can strengthen their quantitative reasoning skills[9, 8, 10]. Paraphrasing Heron and Meltzer, learning to approach problems in a systematic way starts from learning the interrelationships among conceptual knowledge, mathematical skills and logical reasoning.[11] In physics, this necessarily requires the teaching of a good deal of mathematical computations, and dimensional analysis could be used to guide physical intuition.

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