

RIGIDITY OF GEOMETRIC ACTIONS OF NONCOMPACT SEMISIMPLE LIE GROUPS

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In this text I summarize some results of recent years relating to actions of noncompact semisimple Lie groups which preserve geometric actions on manifolds. The results mentioned are generally outgrowths of Robert Zimmer's superrigidity of cocycles and its geometric formulations.

Our general set-up is as follows. Consider a group G acting on a manifold M , preserving some geometric structure ω . The aim is to relate the nature of the structure ω , the topology of M , the group structure (typically some aspect of the representation theory) of G , and various properties of the action, such as fixed points or the nature of the orbits. The relations obtained tend to be of the form of necessary conditions on one of these elements in terms of the others, i.e. obstructions to the existence of the set-up described. Alternatively, we can step back and use such results to relate properties of a structured manifold (M, ω) to properties of the group $Aut(M, \omega)$ of automorphisms preserving the structure.

The techniques used generally yield results under the following conditions.

- (1) G is a semisimple (or sometimes simple) Lie group without compact factors. Often G must be of higher (\mathbb{R} -)rank. In other cases, it must be Kazhdan (have property (T)), a representation-theoretic property which applies when all factors are either higher rank or are the isometry groups of quaternionic hyperbolic space or the Cayley plane.
- (2) M has a finite G -invariant measure, and in some cases must be compact.
- (3) The structure ω is algebraic, i.e. represented by a principal H -bundle on which G acts, where H is an algebraic subgroup of GL_n . In many cases we also allow such subbundles of a higher-order frame bundle. Several results are known for rigid A-structures of Gromov, where some fixed k -jet of local diffeomorphisms actually determines the local diffeomorphism. Often we require H be unimodular, i.e. a (finite) volume density to be preserved.

The standard example of M is a homogeneous space of G .

Another case of interest is where we are concerned with an action of a lattice Γ in G rather than that of G itself as above. Results relate to extending this action to all of G ,

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and hence can be viewed as a geometrization (in the use of a principal H -bundle over M rather than just a single target group H) of Mostow-Margulis superrigidity.

MEASURABLE TRIVIALIZATION AND ALGEBRAIC HULLS

The principal approach has been to measurably trivialize the action on the principal bundle given by the structure. This gives the action in terms of a measurable cocycle $\alpha : M \times G \rightarrow H$. Now ergodic theory can be used to obtain properties of this cocycle. There is an important underlying dichotomy here: An algebraic subgroup $G_1 < G$ of course acts on quotients G/G_2 where G_2 is another closed subgroup. If G_2 is algebraic, then orbits are locally closed (called “tame”) while if G_2 is a lattice (whose algebraic closure is then all of G by Borel Density), then the action is properly ergodic. (By Moore ergodicity, the same holds with G_1 and G_2 reversed)

If the action of (a simple) G on the principal H -bundle is ergodic, an important invariant is the algebraic hull of the cocycle/action, which is the (conjugacy class of) the minimal algebraic subgroup L of H such that the cocycle is measurably equivalent to one taking values in L . If G acts transitively on M , the algebraic hull is just the algebraic closure of the image of the stabilizer under the isotropy representation. One main tool is the Geometric Borel Density Theorem, which states that if H is unimodular, then the Lie algebra of G embeds into that of the algebraic hull L . In this embedding, the action on the tangent space to the orbits results in $Ad\mathfrak{g}$ being a direct summand of this embedding of \mathfrak{g} into GL_n (generalizing the transitive situation).

SUPERRIGIDITY FOR COCYCLES

A generalization of the ideas in Margulis superrigidity gives a corresponding result for cocycles:

Cocycle superrigidity (Zimmer). *Suppose G is Kazhdan and that G or $\Gamma < G$ acts ergodically on a principal H -bundle preserving a finite measure on M . Take H to be a product of connected algebraic groups over local fields of characteristic 0. If the algebraic hull is (alg) connected, then the cocycle α is equivalent to a cocycle $\alpha'(g, m) = \theta(g)c(g, m)$ where the image of c is compact and θ is a homomorphism of the full group $G \rightarrow H$. θ appears in the case H is a real or complex algebraic group. In the p -adic case, we just have c .*

The basic dichotomy between compact closure and homomorphism from G is true for non-Kazhdan higher-rank group. The above precise conclusion is slightly later, as is inclusion by Zimmer and Corlette of the rank 1 Kazhdan groups. Cocycle rigidity can also be expressed geometrically in language of π -simple sections of bundles.

As in the p -adic case, if the destination H were amenable instead of (say) simple algebraic, the cocycle would have compact image. There are other results about various types of cocycles of suitable groups having compact image (Adams, tree results, etc.) The image of the cocycle being compact is equivalent to there being a G - (or Γ -)invariant measurable Riemannian metric.

It is a bit curious that the p -adic case of superrigidity is sometimes used for purely real results, typically via a restriction of scalars argument, looking locally at each prime, eg. to show for a lattice $\Gamma \rightarrow SL_n(\mathbb{Q})$ means the image is virtually in $SL_n(\mathbb{Z})$.

Superrigidity can be used to give a strong result on the nature of the algebraic hull, namely, returning to G higher rank and H real algebraic, then the algebraic hull of the action (of G or of $\Gamma < G$) is reductive with compact center. [This somehow subsumes superrigidity for this situation and somehow combines with it... don't quite understand this philosophically]. The real content is in the unipotent part going away, not the compact center, which follows from Kazhdan and amenability. In order to apply superrigidity, one needs the boundedness arising from the smoothness of the action: purely measurable results have so far only been obtained under restrictive hypotheses (Stuck).

PASSING BACK TO SMOOTHNESS

A significant difficulty using these ideas has been attempting to pass back from “measurable” geometric conclusions which come out of the results on the measurable cocycle structure, to “smooth” geometric conclusions. For example, if we conclude the existence of a measurable invariant Riemannian metric, when can we conclude the existence of a smooth one? Philosophically, one can look at this as sharpening the measurable classification results of actions which fall out to a smooth classification.

I gather this still has many open questions. One known result: If M is compact and a group acts preserving a structure of finite type and measurable invariant metrics on M and on all higher frame bundles $P^{(r)}$ (up to some finite r where there is an $Aut(\omega)$ -invariant framing since finite type), then there is a smooth invariant metric. In particular, this applies (finite type 1) if a connection is preserved together with a measurable metric on M .

[There is a paper by Feres that I just noticed which says something like the existence of a measurable framing for a rigid A-structure actually gives a smooth framing almost everywhere...look at this?]

MORE ON ACTIONS OF LATTICES

Geometric Borel Density tells us that in our set-up (with a volume density being preserved) there is a local embedding $G \rightarrow H$, more specifically into the algebraic hull. It is natural to extend this as far as possible to actions of a lattice Γ .

First, Iozzi has shown that if we have a measurable cocycle from G (no rank limitations) into H then its algebraic hull is the same as that of its restriction to Γ . Her proof makes the assumption that there are no locally closed orbits with compact stabilizers, but she believes that this hypothesis may be unnecessary if one assumes the cocycle arises from a smooth action.

Her result in particular shows that Γ -invariant sections of H -associated bundles with algebraic fibres are also G -invariant. (Here we take the H -structure to be acted on by all of G and Γ is irreducible). This can be taken as another geometrization of the Borel

Density Theorem. This in turn implies that a priori Γ -invariant H -structures are actually G -invariant.

In a different direction, we see from superrigidity for cocycles that even in the measurable case instead of $G \rightarrow H$ locally, we can have a compact cocycle image and hence a measurable Riemannian metric. Zimmer has conjectured that anytime Γ acts smoothly (and G is higher rank) preserving an H -structure which defines a volume density, then if this local embedding does not exist there is an Γ -invariant smooth metric. Using measurable-vs-smoothness ideas similar to those mentioned above, this is known for H of finite type, elliptic, or distal (reductive part is compact). [Any more progress?]

PERTURBATIONS

The study of Γ actions preserving an H -structure on M is a special case of understanding homomorphisms $\Gamma \rightarrow \text{Diff}(M)$. What happens if one perturbs a suitable action a small amount? The ideas under discussion can show that if the action is isometric, under small perturbations which are ergodic and preserve a smooth volume density, one retains the isometric property. Jerome has used differential geometric methods to show that if Γ is cocompact, then this rigidity (remaining isometric) remains true even if the volume preserving and ergodicity hypotheses are a priori removed. This can be seen as a special infinite-dimensional-representational generalization of similar older results with regards to rigidity of finite dimensional representations of Γ . [I really don't understand the techniques in his proof]

Jerome also shows that similar rigidity does not occur too much more generally; in particular, one cannot replace the isometry condition with purely preservation of a rigid A -structure. He gives examples of deformable actions preserving such a structure and a measure (though not a volume) or a volume (but then not the structure). [What is his remark: vol preserving actions of a ss group should be close to algebraic, ie. to actions on homogeneous spaces of the group].

STABILIZERS OF ACTIONS

We can obtain information about properties of possible actions of G on M . The Borel Density Theorem gives us that for any noncompact connected simple G , on each ergodic component, the stabilizer of the action will be either be all of G or discrete a.e. Use of Kazhdan's property lets one sharpen this to saying that for a faithful, irreducible (and properly ergodic) action of a higher rank G , almost all stabilizers trivial. The first result is actually a step in the proof of Geometric Borel Density. Szabo says (after Spatzier?) that for many rigidity-type results it is important to revise the a.e. stabilizers are discrete result to all stabilizers are discrete (locally free action)

There is a suspicion [Gromov-d'Ambra, or I may be confused here, more or less quoting Szabo] that if some sufficiently rigid A -structure is preserved, there are no fixed points. Szabo does this for an affine connection. A remark of Z's: in fact, it is possible that preservation of a volume is really enough for local freeness.

REPRESENTATIONS OF FUNDAMENTAL GROUPS

These ideas (cocycle superrigidity, etc.) can be used to link the representation theory of G acting on M preserving a finite measure and that of $\pi_1(M)$ (no geometric structure around here). The conclusions are that homomorphisms of π_M into another Lie group G' can be nontrivial (not finite) only if there are nontrivial representations $G \rightarrow G'$. In particular, if G has no low dimensional real representations, neither has $\pi_1(M)$. This is yet another generalization of Margulis superrigidity (where $M = G/\Gamma$ so $\pi_1(M) = \Gamma$).

The principle here is to look at the action of the universal cover of G on covers of M . \tilde{M} can then be seen as a $\pi(M)$ -principal \tilde{G} bundle over M and cocycle rigidity exploited. The proof requires assumptions to get suitable ergodicity properties in this bundle, such as “engagement”, where all invariant functions on covers of M must arise from invariant functions on M itself. This seems to cover all known cases, and may not be an essential hypothesis. By work of Gromov a hypothesis of this type is satisfied if everything in sight is analytic. On the other hand, Jerome’s examples give nonengaging actions but the deformations no longer preserve the volume so they do not immediately fall in the scope of this exact question.

Some extensions can be made to actions of lattices of G rather than G itself, but there are difficulties setting up the bundle picture (in particular the liftings to covers) except under restrictive hypotheses.

On a different tack, combining superrigidity with Ratner’s theorem gives that in the higher rank case, if there is an engaging or analytic connection- and volume- preserving action, and $\pi_1(M)$ is a discrete subgroup of a Lie group, then it contains a lattice of another Lie group locally admitting an embedding of G . Without Ratner’s theorem, with additional hypotheses one can obtain the same result if there is a priori some faithful finite dimensional rational representation of π_1 with discrete image.

STUFF OMITTED

I’m omitting three big chunks of ideas in this summary, primarily because I’m not that familiar with them. First are applications of these ideas to orbit equivalence and purely ergodic results, or using “ergodic” ideas of entropy or spectra of unitary representations.

Second, I have been neglecting *specific* geometric consequences of these general ideas, such as structure of the automorphism group of Lorentz manifolds, nonexistence of compact forms of nonriemannian homogeneous spaces, etc. I guess stuff like this is really the *external* motivation for this field, though.

Third, arithmeticity. Isolated question: the extensions of superrigidity to maps from the commensurator of lattices: do these have other known applications other than Margulis arithmeticity theorem?

A fourth smaller chunk: reinterpretation of the cocycle rigidity results in terms of geometric language as opposed to cocycles, ie. π -simple framings in Z’s course notes. Maybe this leads naturally into Feres’ paper of linearizing actions?

A fifth bit: Jerome’s “return” to differential-geometric (and cohomological?) methods to prove his thesis problem. These are completely unfamiliar, maybe I should know something

about them. Also the more functional-analytic arguments involved eg. in amenability of boundary actions [SZ]?

Finally, there are other possible representation destinations that have been considered, for instance representations into automorphism groups of trees?