

SECOND TOPIC PROPOSAL: RIGIDITY IN SYMMETRIC SPACES

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This second topic will deal with some forms of rigidity of lattices in semisimple Lie groups with no compact factors, equivalently finite volume quotients of symmetric spaces of noncompact type. These are the symmetric spaces of nonpositive curvature with no Euclidean (de Rham) factors. The emphasis is on groups/spaces of higher (\mathbb{R} -)rank. See below for a brief discussion of the geometry of these spaces and definition of “rank”. Unless otherwise specified, all Lie groups G and symmetric spaces X in this proposal are assumed to have the properties in the first sentence of this paragraph. We also assume G to be centre-free (though in many cases finite centre is sufficient)

MOSTOW RIGIDITY

Theorem (Mostow Rigidity). *Suppose $\pi : \Gamma_1 \rightarrow \Gamma_2$ is an isomorphism between irreducible lattices in G_1 and G_2 . Then π extends to an isomorphism $G_1 \rightarrow G_2$. In particular, if M_1 and M_2 are two locally symmetric spaces with the same finite volume and isomorphic fundamental group, then they are isometric. We exclude here the exceptional case $G_i = \mathrm{PSL}_2(\mathbb{R})$.*

This result for real hyperbolic manifolds formed the main part of my first topic. The second topic will deal with two generalizations.

Mostow’s proof reduces to considering a Γ -equivariant quasi-isometry (q.i.)² between symmetric spaces. It suffices to show that this q.i. lies within a bounded distance of an isometry. This is done by showing it extends to a map “at infinity”, where there is additional structure which makes it more rigid. The first generalization we consider looks at rigidity of self-q.i.’s of symmetric spaces, dropping the equivariance condition, and q.i.’s of the lattices themselves. Alternatively, one can study extending arbitrary homomorphisms of lattices into (semisimple...) Lie groups. Here, interplay of ideas from algebraic groups and ergodic theory lead to Margulis superrigidity, which subsumes Mostow rigidity. This second generalization is discussed further below. Ergodic techniques are likely to have further applications in the q.i. direction as well.

GEOMETRY OF SYMMETRIC SPACES

We let $X = G/K$ where K is a maximal compact subgroup. Isometrically embedded copies of \mathbb{R}^k are called k -flats. The maximal k such that \mathbb{R}^k embeds is the rank r of X ; r -flats are just called flats and exist through any pointed geodesic. Certain pointed geodesics do not lie in a unique flat; the set of these geodesics divides, by lower-dimensional “walls”, any pointed flat into finitely many

¹First topic, *Kleinian Groups and Mostow Rigidity* [for \mathbb{RH}^n] taken Mar. 95, examiners Robert Zimmer and Benson Farb.

²Two spaces are quasi-isometric if they are equivalent in the category of coarse bilipshitz maps.

isometric sectors called Weyl chambers. Let A^+ be a chosen Weyl chamber; we obtain a polar decomposition $K/\text{Fix}(A^+) \times \overline{A^+} \rightarrow X$.³ The left factor is analogous to the visual angle.

Since X is nonpositively curved, we can form a boundary at infinity $\partial_\infty X$ of asymptote classes of directed geodesics. The Tits angle between two ideal points is the supremal angle subtended by intersecting geodesics representing the two points. The Tits boundary is $\partial_\infty X$ equipped with the length metric induced by this angle. For instance, ideal points of a flat have the usual angle distance, while ideal points in the image of an isometric embedding $\mathbb{H}^k \rightarrow X$ have distance ∞ since they are joined by a geodesic.

Chambers and walls project naturally to the Tits boundary. The images of flats are isometrically embedded copies of S^{r-1} , called apartments, consisting of chambers (sectors) separated by walls. Furthermore, these subdivisions are maintained under a choice of a different pointed flats, and any two points of $\partial_\infty X$ lie in an apartment. This structure is called the Tits building of X . This structure on $\partial_\infty X$ provides the ‘‘rigidity at infinity’’ argument used for Mostow’s theorem. The main ingredient is a theorem of Tits: provided there are no rank 1 factors, an isomorphism of the Tits buildings arises from an isometry of spaces.

The above results follow most easily from analysing the Lie algebra structure of G , and will be looked at as a part of this topic. It is possible to define rank, chambers, and obtain a Tits structure using purely differential geometric arguments, for instance on the dimension of the space of parallel Jacobi fields. This allows [BGS,Eb] for Mostow-type rigidity results for certain spaces which are not a priori symmetric; however the results also show that such (de Rham) irreducible spaces of generalized rank > 1 are in fact symmetric. These results are beyond the scope of this topic, but open up interesting more general rigidity questions.

Q.I. RIGIDITY RESULTS

For any space S , let $\text{QI}(S)$ be the group of self-q.i.’s modulo equivalence (roughly up to bounded sup norm). Call a family of spaces \mathcal{S} q.i. rigid if \mathcal{S} is closed under q.i.’s (in some implied larger class of spaces) and is minimal as such. We say two subgroups of a group C are commensurable if they have common finite index subgroups. Denote by $\text{Comm}(D)$ the commensurator of a subgroup D , ie. the set of $c \in C$ such that D and $c^{-1}Dc$ are commensurable. More generally, two groups are abstractly commensurable if they have isomorphic finite index subgroups.

Consider the following claims, which should hold for a wide class of Lie groups and spaces. We discuss below conditions under which this has been proven.

Claim 1. *Any q.i. of X is a bounded distance from an isometry, ie. the natural map $\text{Isom}(X) \rightarrow \text{QI}(X)$ is an isomorphism.*

Corollary 1a. *Finite extensions of cocompact lattices in G form a q.i. rigid family among all finitely generated groups (being exactly those q.i. to G and X).*

Corollary 1b. *If X_1 and X_2 are quasi-isometric, then X_1 and X_2 are isometric up to (de Rham) factor rescaling.*

Claim 2. *Given a nonuniform (ie. non-cocompact) irreducible lattice Γ in G , the natural map $\text{Comm}(\Gamma) \rightarrow \text{QI}(\Gamma)$ is an isomorphism.*

³Elements of $\text{Fix}(A^+)$ map points in A^+ into A^+ .

Corollary 2a. *The commensurability classes of finite extensions of nonuniform irreducible lattices in G form q.i.-rigid families among all finitely generated groups.*

Corollaries 1a and 2a form a part of Gromov’s program ([Gro]) of classifying finitely generated groups up to quasi-isometry. Claim 1 implies Mostow.

Considerable progress in proving these claims has been made in the past few years. Claim 1 was shown for the rank one spaces $\mathbb{H}\mathbb{H}^n$ and $Ca\mathbb{H}^2$ by Pansu [P]; it is false for $\mathbb{R}\mathbb{H}^n$ and $\mathbb{C}\mathbb{H}^n$. Recently it was shown true for higher rank spaces with no rank one de Rham factors in [KL] and [EF]. A study of the proof techniques in the latter two papers forms an integral part of this topic.

Claim 2 has been proven in [S] for the rank 1 case (it is false for $\mathbb{R}\mathbb{H}^2$), and the proof extended to products of several copies of $\mathbb{R}\mathbb{H}^2$ in [FS] and by further work of Schwartz. Recent work of Eskin [E] proves it for the higher rank case with no rank 1 factors.

QUASI-RIGIDITY AT INFINITY

The lack of equivariance in Claim 1 above makes it harder to show that the Tits building at infinity is preserved. One must deal with the coarseness of the map without additional tools.

The approach of Kleiner and Leeb is to consider the asymptotic cone $\text{Cone}(X)$, which is the (ultra)limit of the rescaled spaces $(X, d/\lambda_i)$ where $\lambda_i \rightarrow \infty$. This is loosely the space we see in the limit as we fly away rapidly from X . It is clear that coning smooths out coarseness and emphasizes asymptotic behaviour. For instance, $\text{Cone}(\mathbb{Z}^n) = \text{Cone}(\mathbb{R}^n) = \mathbb{R}^n$. In contrast, the cone of a strictly negatively curved space is an \mathbb{R} -tree (this requires nonprincipal ultrafilters and hence the axiom of choice). The Tits building structure essentially carries over to the cone and the smoothing out of the quasi-isometry is sufficient to give rigidity.

Eskin and Farb instead generalize a lemma of Mostow stating that a q.i. embedding of \mathbb{R} into $\mathbb{R}\mathbb{H}^n$ is within a bounded distance of a geodesic.

Theorem (Eskin-Farb). *A q.i. embedding of \mathbb{R}^r into X lies within a finite distance of a (uniformly) finite number of flats.*

Their approach uses the Furstenberg boundary \hat{X} of X , which is obtained by collapsing each chamber at infinity in $\partial_\infty X$ to a point, so that the polar decomposition of X is $\hat{X} \times \overline{A^+}$ and the points in \hat{X} are “visual angles”. The result follows from showing that (loosely) the image of \mathbb{R}^r projects to a finite number of points in \hat{X} . Pairs of points which are “opposite” span a unique flat (as opposed to a horospherically-indexed family) in X and the image of \mathbb{R}^r lies close to the union of these flats. Two main difficulties arise. First, the visual angle is not well defined on $\hat{X} \times (\partial\overline{A^+})$ so one must consider only “nondegenerate” points in \mathbb{R}^r which embed away from these directions. Second, it is surprisingly difficult to show that the visual angle spanned by regions of such points actually converges to a point.

If the q.i. embedding in the theorem is a restriction of a q.i. of X , then one can see that the image of a flat actually lies close to a single flat. Moreover, chambers and walls in $\partial_\infty X$ go to chambers and walls, and so rigidity follows as in the Mostow case.

This topic will include understanding these arguments as they appear in [EF] and [KL].

NONUNIFORM LATTICES

The q.i. rigidity of nonuniform lattices is considerably more difficult to deal with. An irreducible nonuniform lattice Γ is not q.i. to X ; however, by slicing off the cusps at infinity it is q.i. to X with a Γ -equivariant union of horoballs removed, called a neutered space. The large-scale geometry of Γ lies in the packing of this union of horoballs.

If X has rank 1 or is a product of \mathbb{RH}^2 's, the removed horoballs can be made disjoint, so the boundary of the neutered space consists of horospheres wedged into $\partial_\infty X$. By the work of Schwartz and Farb, a q.i. of Γ arises from a q.i. of this neutered space which preserves the components of this boundary. Furthermore, the induced map on the internal boundary of each horosphere is bilipshitz, and an argument in the spirit of Tukia ([T]) for \mathbb{RH}^n (discussed in my first topic) shows that the map must be the identity on the boundary up to conjugation. Considerable large-scale-topological and number-theoretic (we are dealing with the Hilbert modular groups) complications arise.

Unfortunately in the general case, the horoballs removed may not be disjoint and so this argument fails. However, the theorem of Eskin-Farb in the previous section extends to embeddings of flats with reasonable holes removed. An argument of Eskin [E], will use this together with Moore Ergodicity (see below) to prove claim 2 for spaces X with no rank 1 factors. It seems these techniques may improve the results of Schwartz and Farb and could lead to a complete resolution of claim 2.

The results in this section will not be emphasized in this topic though they do form a possible direction for further study.

MOORE ERGODICITY

Ergodic theory is the study of the qualitative nature of orbits under a group action G on a measure space S . We assume that null measure of sets is preserved (quasi-invariant measure). We say the action is ergodic if G -invariant sets are either null or conull. In particular this means G -invariant measurable functions into nice spaces (eg. \mathbb{R}) are a.e. constant. The action is tame if orbits are locally closed. If both ergodicity and tameness occur, the measure is supported on an orbit. If not supported on an orbit, the action is said to be properly ergodic and orbits are dense.

Theorem (Moore Ergodicity). *Let G be a semisimple Lie group, Γ an irreducible lattice, and H a closed subgroup. Then the actions of H on G/Γ and Γ on G/H are ergodic iff H is noncompact. (The measures belong to the unique G -invariant measure class).*

For instance this means ergodicity of the action at infinity of a lattice Γ in $\text{Isom}(X)$, for the boundary arises as the quotient of G by the parabolic subgroup fixing a point at infinity. This gives an ingredient for certain rigidity proofs; for instance, for Mostow rigidity in \mathbb{RH}^n , the induced map at infinity is shown to be quasi-conformal. By ergodicity, the eccentricity of the distortion field is constant, which leads to showing the map is actually conformal.

Moore's theorem has an stronger formulation in terms of the vanishing of matrix coefficients of unitary representations of G as one leaves compact sets. This result is proved in [Zim, Chapter 2], which is a part of this topic.

MARGULIS SUPERRIGIDITY

It can be shown that a connected semisimple Lie group can essentially be embedded as the real points $J_{\mathbb{R}}^0$ of a semi-simple algebraic subgroup J of $\text{GL}_n(\mathbb{C})$ defined over \mathbb{Q} . Furthermore, group actions of algebraic groups on varieties are tame (in the sense of the previous section). The interplay of ergodicity and tameness leads to a proof of

Theorem (Margulis superrigidity). *Suppose G is connected semisimple algebraic group of rank > 1 , and $G_{\mathbb{R}}^0$ has no compact factors. Suppose $\pi : \Gamma \rightarrow H$ is a homomorphism of an irreducible lattice in G to a simple algebraic group H (centre-free) with $H_{\mathbb{R}}$ noncompact. If the image is Zariski dense, then π extends to a rational homomorphism $G_{\mathbb{R}}^0 \rightarrow H$ over \mathbb{R} (giving a homomorphism $G_{\mathbb{R}} \rightarrow H_{\mathbb{R}}$).*

Superrigidity implies Mostow rigidity (for rank > 1) because of the following result, which also plays an important role in the proof of superrigidity.

Theorem (Borel Density). *Suppose G is connected semisimple algebraic group and $G_{\mathbb{R}}^0$ has no compact factors. If Γ is a closed cofinite volume subgroup, then it is Zariski dense in G .*

To prove superrigidity, it suffices to find a (measurable) Γ -map $\hat{\phi} : G_{\mathbb{R}}^0/P_0 \rightarrow H_{\mathbb{R}}/L_{\mathbb{R}}$ for a minimal parabolic P and some closed algebraic subgroup L , which one can show must be a.e. rational. This “boundary map” then induces the required isomorphism extending π via a Zariski closure argument. To get $\hat{\phi}$, we see that the action of Γ on $G_{\mathbb{R}}^0/P_0$ is “amenable”, so there is a Γ -map ϕ into the space of probability measures \mathcal{M} on $H_{\mathbb{R}}/Q_{\mathbb{R}}$ where Q is a parabolic subgroup. The action of $H_{\mathbb{R}}$ on \mathcal{M} is tame and the composition ϕ modulo $H_{\mathbb{R}}$ is a.e. constant by Moore Ergodicity. Thus ϕ can be viewed as a map $\hat{\phi}$ into $H_{\mathbb{R}}/L_{\mathbb{R}}$, where $L_{\mathbb{R}}$ is the stabilizer of the $H_{\mathbb{R}}$ -orbit.

This proof appears in [Zim, Chapter 5]. Together with Chapter 4 (Amenability) and a general overview of Chapter 3 (Algebraic groups) it will form the final part of this second topic.

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