KLEINIAN GROUPS AND MOSTOW RIGIDITY: FIRST TOPIC PROPOSAL

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Let M_1 and M_2 be two hyperbolic n-manifolds of finite volume, where $n \geq 3$. Mostow's Rigidity Theorem states that whenever M_1 and M_2 have the same fundamental group, then they are isometric. This remarkable result links together concepts from topology, geometry, and group theory: the fundamental groups of M_1 and M_2 arise naturally as lattices in the isometry group of hyperbolic space $\mathbb{H}^n$. In this formulation, the rigidity result takes the form that two such isomorphic lattices are actually conjugate.

The proof of Mostow's Theorem uses important concepts from the study of discontinuously acting groups of hyperbolic isometries, which are called Kleinian groups. Basic ideas in the proof have had a considerable influence on current large-scale geometrical thinking. The content of this first topic will be to present Mostow's Theorem, its proof, and some generalizations, using it as a vehicle for exploring these concepts and ideas.

It is anticipated that my second topic will deal with lattices in more general Lie groups and with geometric group theory/large scale geometry. Kleinian groups form a core class of examples here, and the first topic will be constructed with these prospects in mind.

Hyperbolic space and the Möbius group. The Möbius group $\text{Möb}(n)$ is generated by reflections in planes and spheres (inversions) in $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\} \approx S^n$. By Liouville's theorem, these are exactly the conformal automorphisms of $\overline{\mathbb{R}}^n$.

The natural embeddings of $\overline{\mathbb{R}}^n$ or S^n in $\overline{\mathbb{R}}^{n+1}$ lead to embeddings of $\text{Möb}(n)$ in $\text{Möb}(n+1)$ by reflecting in planes (spheres) perpendicular to the embedded hypersurface. The extended transformations exactly preserve the Poincaré hyperbolic metric in H^{n+1} or B^{n+1}. Conversely, isometries of $\mathbb{H}^{n+1}$ extend to conformal automorphisms of $\overline{\mathbb{R}}^n$, considered as "the sphere at infinity". This leads to the relationship $\text{Möb}(n) = \text{Isom}(\mathbb{H}^{n+1}) < \text{Möb}(n+1)$.

One can consider the one-sheet hyperboloid model of $\mathbb{H}^{n+1}$ to obtain $\text{Möb}(n) \simeq \text{SO}(1,n)/\{\pm 1\}$. This representation gives one of several equivalent ways to topologize the Möbius group.

Kleinian subgroups and hyperbolic manifolds. Let G be a subgroup of $\text{Möb}(n)$. The limit set $L(G)$ is the set of limit points of all orbits (in $\overline{\mathbb{R}}^n$). It is closed and G-invariant. The complement is the discontinuity set $\Omega(G)$. We call G Kleinian if it acts properly discontinuously in $\mathbb{H}^{n+1}$; it is then discrete. Under the inclusion $G < \text{Möb}(n) < \text{Möb}(n+1)$, we then have $L_{\overline{\mathbb{R}}^n}(G) = L_{\overline{\mathbb{R}}^{n+1}}(G) \subseteq \partial\mathbb{H}^{n+1}$. Henceforth we let G denote a Kleinian group. If G is fixed point free, $\Omega(G)/G$ is a conformal manifold and $\mathbb{H}^{n+1}/G$ is a hyperbolic manifold[1] . Both have fundamental group G.

[1] M is an *adjective* manifold if it has an *adjective* structure on it; roughly, if the transition functions preserve *adjective* characteristics.

A classical result of Klein-Poincaré states that all complete 2-manifolds are Euclidean, conformal, or hyperbolic, i.e., arise as quotients by a discrete group of isometries of $\mathbb{R}^2$, $\overline{\mathbb{R}}^2$, or $\mathbb{H}^2$ respectively, with the hyperbolic case being the generic case. In dimension 3, the Geometrization Conjecture of Thurston states that a compact manifold is glued together from pieces X/G, where G is a discontinuous group of isometries of a geometric 3-space X. The generic case is again $X = \mathbb{H}^3$; apart from Euclidean and conformal structures there are 5 more exceptional geometries.

These two results provide important motivation for the topological importance of Kleinian groups. However, the techniques involved are quite specialized. In the topic, I will concentrate on approaches meaningful in higher dimension and will treat these results as a black box.

Lattices. A particularly important case occurs when $\mathbb{H}^n/G$ has finite volume. Then we say $G = \Gamma$ is a lattice. There exists a closed set $F \subset \mathbb{H}^n$ whose images under Γ cover $\mathbb{H}^n$ such that the interiors are disjoint. F is called a fundamental domain for Γ and $F/\Gamma \approx \mathbb{H}^n/\Gamma$ as orbifolds. The boundary of F can be taken to consist of hyperbolic k-planes $(0 \leq k < n)$ called faces. The lattice Γ is generated by those elements which pair the codimension 1 faces. Additional relations are given by the manner in which images of F stack up around the other faces of F. The "itinerary" of images of F visited along a closed loop in $\mathbb{H}^n$ modulo these relations gives the class of the loop in $\pi_1(\mathbb{H}^n/\Gamma) \simeq \Gamma$.

Mostow's Rigidity Theorem. Recall from above that this theorem states that two lattices in $\mathrm{Isom}(\mathbb{H}^n)$ for $n \geq 3$ are isomorphic only if they are actually conjugate. Actually, the isometry corresponding to a chosen isomorphism is unique. Mostow's Theorem is false for $n = 2$.

The proof of the Theorem has the following basic steps. First, a map ϕ between metric spaces E and F is called a quasi-isometry if ϕ is bilipshitz for pairs of points sufficiently distant. If there is a quasi-isometry $\psi: F \to E$ such that $d(\phi\psi, \mathrm{Id})$ and $d(\psi\phi, \mathrm{Id})$ are bounded, E and F are said to be quasi-isometric. In Mostow's theorem, since manifolds are of type $K(\pi, 1)$, an isomorphism of first fundamental groups arises from a homotopy equivalence $f: M_1 \to M_2$, $g: M_2 \to M_1$. We consider the lifts $\tilde{f}$ and $\tilde{g}$ to $\mathbb{H}^n$. The first step is to show that these lifts are quasi-isometries. This is not difficult in the compact case.

The second step consists of showing $\tilde{f}$ and $\tilde{g}$ extend continuously to maps of the sphere at infinity. Furthermore, the extensions are quasi-conformal (in the usual sense) there. This can be done by ad-hoc geometrical reasoning.

The third step is to show that these quasi-conformal extensions are actually conformal and hence $\tilde{f}$ and $\tilde{g}$ are (can be taken to be) hyperbolic isometries. We discuss two approaches to this step below.

We see two particularly significant ideas present in this proof outline. First, there exist the coarse concepts of 'quasi-isometry' and 'quasi-conformality at infinity', which interact similarly as do the concepts of 'isometry' and 'conformality on the boundary'. Second, a rather refined understanding of the nature of the conformal action at infinity of a hyperbolic lattice allows us to pass back from the coarser concepts to the desired finer ones. With this in mind, we discuss Möbius transformations in a bit more detail.

Structure of a Möbius transformation. There is an important trichotomy of $\mathrm{M\ddot{o}b}(n)$ according to the fixed points of the extended action in $\overline{\mathbb{H}}^{n+1}$. If g has a unique fixed point which lies in $\partial\mathbb{H}^{n+1}$, it is called parabolic. Taking (up to conjugation) the fixed point to be ∞ in the half-space model, g takes the form $g(x) = A(x) + a$, with a not orthogonal to the eigenspace of A, on $\overline{\mathbb{R}}^n$. If there are just two fixed points, also on the boundary, then g is called loxodromic. Up to conjugation, $g(x) = rA(x)$. Otherwise, there are fixed points inside $\mathbb{H}^{n+1}$; such g are called elliptic. Many arguments are simplified by conjugating a given transformation to the canonical form as above for its fixed point type.

Inside $\mathbb{H}^{n+1}$, we consider the displacement function $\Delta_g(x) = d_{\mathbb{H}}(x, g(x))$. According as g is elliptic, loxodromic, or parabolic, Δ_g has minimum zero attained, positive minimum attained on the geodesic joining the fixed points (axis), or no minimum. The names of the types arise from considering the foliations of $\mathbb{H}^n$ such transformations leave invariant; for instance, for parabolic g, the leaves are horospheres through the fixed point.

Let Γ be a lattice in $\mathrm{Isom}(\mathbb{H}^{n+1})$ with fundamental region F. It is seen that $\overline{F}$ cannot contain loxodromic fixed points (i.e., Γ cannot contain loxodromic elements) and it contains parabolic fixed points only if Γ is not cocompact. In this case, the cusps of $\mathbb{H}^{n+1}/\Gamma$ occur at these parabolic fixed points, near which exist arbitrarily short homotopically nontrivial loops (cusps are "thin"). By chopping off the cusps, we obtain a compact "thick" manifold.

If $\mathcal{H}$ is an arbitrary nonpositively curved simply connected manifold, the boundary $\partial\mathcal{H}$ can be defined via equivalence classes of asymptotic geodesics. Under suitable conditions, there is a unique geodesic through any pair of finite or infinite points. Isometries of $\mathcal{H}$ can be categorized as for $\mathbb{H}^n$ via the displacement function or by fixed point type in $\mathcal{H} \cup \partial\mathcal{H}$. Quotients of $\mathcal{H}$ with this unique geodesic propoerty by discrete groups of isometries are called visibility manifolds; their nature is similarly described by this categorization as for hyperbolic manifolds.

The set at infinity. We now examine the action of a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ on the boundary sphere at infinity somewhat more carefully. We have seen that all of the group structure is reflected in this action, via the isomorphism $\mathrm{Isom}(\mathbb{H}^n) \simeq \mathrm{M\ddot{o}b}(n-1)$.

We can formalize this for visibility manifolds V. Topologize $\overline{V} = V \cup \partial V$ with a local basis at $x \in \partial V$ consisting of cones with vertex inside V which "include" x; this topology s coincides with the usual topology on S^{n-1} or $\overline{\mathbb{R}}^n$ for the $\mathbb{H}^n$ case.

If we allow interiors of horoballs as neighbourhoods of points in ∂V, we get a larger topology h. We can also allow cones with vertex at infinity (topology a).[2] We then get a relation among corresponding limit sets, $L_a(\Gamma) \subseteq L_h(\Gamma) \subseteq L_s(\Gamma)$. In particular, loxodromic fixed points lie in L_a, and parabolic fixed points in $L_s \setminus L_h$.

We return to the case of $\mathbb{H}^n$ and the natural topology s there. $L(\Gamma) = L_s(\Gamma)$ is the limit set of any orbit, so in particular is the closure of the set of loxodromic fixed points or the set of parabolic fixed points, if any. In fact, the loxodromic fixed points are dense in $L(\Gamma) \times L(\Gamma)$. Apanasov has shown that for a lattice, $L_a(\Gamma) = L_h(\Gamma)$ and $L = L_s = L_a \cup \{\text{parabolic fixed points}\}$.

More generally, the action of Γ splits the sphere at infinity (up to a null set) into a dissipative part (disjoint subsets permuted by Γ) and a conservative part (infinitely many images of a subset W of positive measure intersect W in a set of positive measure). The measure here is standard Lebesgue measure on the sphere. The conservative part is exactly $L_h(\Gamma)$.

Now let Γ act on $S^{n-1} \times S^{n-1}$ (the space of geodesics). It has been shown (E. Hopf/Sullivan) that $L_a(\Gamma)$ has either 0 or full measure, according as this action is dissipative or ergodic (i.e., measurable invariant subsets of $S^{n-1} \times S^{n-1}$ have 0 or full product measure; invariant measurable functions are a.e. constant). This provides one approach to the third step of the proof of Mostow's Theorem. In this case, the limit set has full measure so the action is ergodic; from this it follows (roughly) that the quasiconformal extension $\tilde{f}$ at infinity has "a.e. constant eccentricity" and hence is conformal.

[2]The proposer has yet to find these topologies discussed together in the literature. Unfortunately, concepts relating to either of the s and a topologies are variously called "conical".

Another approach involves zooming in near a point $x \in L_a$ where $\tilde{f}$ has a nonvanishing Jacobian. Since points of L_a do not arise on the boundary of fundamental regions of lattices, the equivariant map $\tilde{f}$ must exhibit "full complexity" near x, and hence is conformal. One proves these claims via a limiting process along conical neighbourhoods of x.

Large-scale geometry. In the outline of the proof of Mostow's Theorem, we saw the importance of a coarse, large-scale geometrical equivalence, namely quasi-isometry. Now consider an arbitrary finitely generated group G with a given generating set. We equip G with the word metric (the discrete metric given by length of a shortest word representing each element), or alternatively consider the Cayley graph with the path metric. If G acts discontinuously on a geometrical space X, for any fixed $x_0 \in X$ we get an embedding of G in X via the orbit of x_0. This is a quasi-isometry; in particular, G is quasi-isometric to itself equipped with any other generating set.

We get a natural concept of the boundary ∂G by looking at the closure of the Cayley graph (possibly with edge lengths changed to ensure "summability"). If $X = \mathbb{H}^n$ and G acts cocompactly, then $\partial G = L(G)$ in the above embedding. This begins to illustrate how geomerical methods provide techniques for studying finitely generated groups up to quasi-isometry. In recent years, extension of ideas present in the proof of Mostow's Theorem have proved powerful in this study.

References

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