

# RESEARCH SUMMARY

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I study Lie group actions on manifolds preserving geometric structures, using techniques from dynamics and linear algebraic groups. My thesis advisor is Robert Zimmer, and my work forms part of an ongoing program to describe such actions.

## 1. STATEMENT OF RESULTS

Let  $G$  be a real algebraic group. A subgroup  $H < G$  is called *observable* if it is the stabilizer of a vector in some finite-dimensional linear representation of  $G$ , and *epimorphic* if whenever  $H$  fixes such a vector then it is fixed by all of  $G$ . Let us further call  $H$  *connection-observable* if  $H$  is the stabilizer of a point in a  $G$ -action on some manifold  $M$  preserving an affine connection, and *connection-epimorphic* if any point fixed by  $H$  in such a  $G$ -action must be fixed by all of  $G$ .

In my thesis, I state the following conjecture and prove it in certain cases [Per2].

**Conjecture.** *Suppose  $G$  is any real algebraic group, and  $H$  an algebraic (closed) subgroup. Then*

1.  $H < G$  is connection-observable iff it is observable, and
2.  $H < G$  is connection-epimorphic iff it is epimorphic.

**Theorem 1** (Main theorem). *The conjecture is true when  $G$  is reductive and either of the following situations holds:*

- “ $G$  is small”, i.e., the simple factors of  $G/\mathcal{R}G$  are (a) of  $\mathbb{R}$ -rank 1, (b)  $SL_3(\mathbb{R})$ , or (c) compact;
- “ $H$  is big”, i.e.,  $H$  is normalized by a maximal torus  $T$  of  $G$  (part 2 of the conjecture), or  $\mathbb{R}\text{-rank}(H) = \mathbb{R}\text{-rank}(G)$  (part 1).

I have also obtained the following structure result about epimorphic subgroups [Per1]:

**Theorem 2.** *There is a one-to-one correspondence between*

1. minimal algebraic epimorphic subgroups  $H$  of  $SL_n$  normalized by a maximal torus  $T$ , classified up to conjugacy by the Weyl group; and
2. pairs  $(\tau, p)$  of (a) trees  $\tau$  on  $n$  vertices, up to isomorphism, together with (b) points  $p$  in a certain subset of the Tits boundary  $\mathcal{B}$  of  $T$  determined by a canonical embedding of  $\tau$ .

*The unipotent radical of such an  $H$  (which is always solvable) can be read off from a suitable numbering of the vertices of  $\tau$ , and the maximal subtorus is determined by  $p$ .*

## 2. BACKGROUND

Suppose  $G$  acts on a manifold  $M$  preserving some geometric structure  $\omega$ . The Zimmer program [Zim2] aims to relate (1) the group structure of  $G$ , (2) the topology of  $M$ , (3) the nature of the structure  $\omega$ , and (4) the dynamics of the action. Typically,  $G$  is a semisimple Lie group or a discrete subgroup of such. Examples of geometric structures include Riemannian or pseudo-Riemannian metrics, invariant measures, and (as in my work) affine connections. Considerable progress has been made by many researchers under various hypotheses. My results provide information on fixed points and isotropy subgroups.

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In his 1986 I.C.M. address [Zim1], Zimmer described the above program from a slightly different point of view. We generally study a group  $G$  by examining its representations (or realizations) in some natural class  $C$  of groups. For instance, by letting  $C$  consist of endomorphism groups  $GL(V)$  of finite-dimensional vector spaces, we get finite-dimensional linear representation theory. Infinite-dimensional unitary representation theory is another example. In our case, we consider a form of “geometric representation theory”, where we let  $C$  consist of automorphism groups  $\text{Aut}(M, \omega)$  of various geometric structures on manifolds. The aim is, of course, to discover interesting new information on  $G$ , and to uncover parallels between the various representation theories. My results provide one such parallel.

Finally, G. D’Ambra and M. Gromov have stated [DG] the “vague general conjecture” that all triples  $(G, M, \omega)$ , under certain hypotheses, should be “almost” classifiable. Results in this direction have been obtained in a number of special cases, and the study of isotropy has proven crucial.

The notions of observable and epimorphic subgroups were introduced by Bialynicki-Birula, Hochschild, Mostow, and Bergman in the 1960s and 1970s. More recent work has been done by Bien and Borel (e.g., [BB1, BB2]) and Grosshans [Gro]. In particular, Bien and Borel (following Pommerening [Pom]) present necessary and sufficient algebraic conditions for subgroups normalized by a maximal torus in a semisimple group to be epimorphic, and construct one type of example which is minimal (i.e., contains no smaller subgroup which is still epimorphic and normalized). My classification of all such minimal subgroups (Theorem 2) follows from a reinterpretation and generalization of their conditions in terms of graphs and the Tits boundary.

Results somewhat analogous to my Conjecture, but where  $\omega$  is an invariant measure rather than an affine connection, have recently been obtained by S. Mozes [Moz] and Barak Weiss [Wei] for  $M$  a homogeneous space. E. Goetze [Goe], J. Szaro [Sza], and A. Zeghib [Zeg] have used different approaches to obtain restrictions on  $M$  (and also  $G$ ) in connection-preserving actions.

### 3. OUTLINE OF PROOF

**3.1. Techniques.** My proof of Theorem 1 uses a mixture of local dynamics and algebraic techniques. The crucial feature of connection-preserving actions is that the action of an isotropy subgroup  $G_m$  linearizes in some neighbourhood of the fixed point  $m$ . The difference from the linear-representation picture is that the linearization (i) is not global in  $TM_m$ , and (ii) does not extend to all of  $G$ . Since the question is local, there is no need for the ergodic techniques typical of parts of the Zimmer program, and thus hypotheses on the existence of an invariant measure or on  $G$  having no compact factors are not necessary.

**3.2. Observable-epimorphic reduction.** The implications observable  $\Rightarrow$  connection-observable and connection-epimorphic  $\Rightarrow$  epimorphic are trivial. The remaining implication in part 1 of the conjecture is then shown to follow from part 2, using the “oppositeness” of the observability and epimorphicity properties, with some technical complications.

**3.3. The  $SL_2$  local picture.** Playing off the dynamics of the action versus the isotropy linearization and examining the unipotent orbits yields the following

**Lemma.** *Suppose a connection-preserving action of  $SL_2 = KAN$  has the property that  $N$  fixes some  $A$ -orbit pointwise. Then that orbit is a single point which is fixed by all of  $SL_2$ .*

In particular, since  $AN$  is the only epimorphic subgroup of  $SL_2$ , this proves the result for  $SL_2$ .

**3.4. Reduction to  $SL_2$ .** Finally, the general case of part 2 is reduced to the above lemma via a refinement (\*) of the results of Bien and Borel on the structure of epimorphic subgroups. The “big  $H$ ”  $T$ -normalization hypothesis allows this reduction to take place in general, and consideration of one other class of examples provides the result under the “small  $G$ ” hypothesis. It is likely that the “small  $G$ ” argument works without further ingredients for other rank 2 factors.

3.5. **Theorem 2.** This generalizes the refinement (\*) in 3.4, translating conditions on a subgroup  $H < SL_n$  to be epimorphic into edge conditions on a numbered graph  $G$  on  $n$  vertices. Conjugation corresponds to permuting the numbering, and minimality to the graph being a tree. Minimal subtori in  $H$  satisfy an algebraic condition given by the unipotent radical of  $H$ , and this condition translates nicely into the Tits boundary when  $G$  is embedded via the root lattice of  $SL_n$ .

#### 4. FUTURE PLANS

4.1. **Completing the proof of the conjecture.** Some additional technical results on the structure of epimorphic subgroups over  $\mathbb{R}$  should eliminate the need for  $G$  to be reductive, and relax the  $H$  full-rank condition to  $T$ -normalization in part 1 of the conjecture.

It may actually be possible to remove the  $T$ -normalization condition altogether (thus proving the full conjecture) via an extension of the existing argument. One possibility is to refine the structure theory of minimal epimorphic subgroups further, along the lines of [Per1], to still “feed in” to the step 3.3. The difficulty here is that the lack of  $T$ -normalization means that an epimorphic  $H$  no longer contains whole root spaces of  $G$ , which provided the main tool in the Bien-Borel analysis. However, there is a technique of Sukhanov [Suk, Gro] which bypasses  $T$ -normalization of observable subgroups in another context via an induced action and geometric invariant theory argument.

4.2. **Nonalgebraic  $H$ .** What if one removes the requirement that  $H$  be an algebraic subgroup from my definition of connection observable/epimorphic? It is clear that discrete subgroups can arise as point stabilizers, while finite index subgroups of epimorphic subgroups are still epimorphic. It is my belief that this is in some sense the “only” example, and that the Conjecture should hold for all connected subgroups  $H$ . A starting point here is work of D. Witte [Wit] on cocompact subgroups of semisimple groups, since a form of cocompactness is used in the  $SL_2$  analysis in my proof.

4.3. **Other linearizable actions.** The only actual use of the connection in my results is to guarantee local linearization of the isotropy representation (see 3.1). Accordingly, the “local analysis” techniques used should give comparable results for any actions which are linearizable near fixed points. Some conditions under which this is true (conjectured in [DG]) are demonstrated, together with an overview of the problem, in [CG].

4.4. **Invariant submanifolds and subgroup actions.** Finally, any study of fixed points of an action can be viewed as the first step in the study and classification of invariant submanifolds, and of the orbit structure in general. Perhaps some of the ideas in my work can be generalized to give information on higher-dimensional invariant submanifolds under subgroup actions.

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