# EXPONENTIALS AND LOGARITHMS BY CONTINUOUS EXTENSION

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The exponential and logarithm functions are extraordinarily important in one-variable calculus. However, they are hard to teach with any attempt at rigor without completely confusing students. Conventional approaches are either very roundabout, or skip the hard (and interesting) bits altogether. This paper attempts to bridge the gap. In the process, it discusses a fundamental principle underlying the use of graphing calculators (or computers) —"connecting the dots", or continuous extension from a dense subset.

The challenge (as well as the beauty) of the whole subject is that the exponential and logarithm functions[1] play a number of roles in calculus:

(R1) $b^x$ is the extension of the formula $b^{p/q} = \sqrt[q]{b^p}$ to real $x$.
(R2) $e^x$ is its own derivative; $e$ is "some crazy number" which makes this happen.
(R3) $b^x$ is proportional to its own derivative; the proportionality factor is $\ln b$.
(R4) $e^x$ and $\ln x$ are inverses, and $b^x = e^{(\ln b)x}$.
(R5) $\ln x = \int_1^x t^{-1} dt$.
(R6) $e^x$ is the solution to the differential equation $y' = y$, $y(0) = 1$.
(R7) $e^x = \lim_{n\to\infty}(1 + x/n)^n$.

Some calculus texts, for instance [5, 6, 7], define $\ln x$ via (R5) and $e^x$ and $b^x$ via (R4). The other roles become just computational facts. The reason for this roundabout approach is the difficulty of doing (R1) properly. It is hard, however, to explain this difficulty to students, who feel they already know what $b^x$ is, and mathematicians must be crazy if they don't.

In response, other calculus texts, for instance [2, 3]), gloss over (R1) and then empirically observe the first half of (R3), followed by (R2). Sometimes they introduce $e$ by (R7), glossing over convergence. They define ln via (R4), and derive the other roles as necessary. This is certainly much more natural, but foundationally incomplete.

An interesting paper, [8], builds the missing foundation by defining $e^x$ via (R6). The existence and uniqueness of a solution to the differential equation is proven by developing exactly enough about infinite series to show the power series for $e^x$ (yet another role!) converges. In this paper, I follow an alternate route, starting with (R1) and (R3). Power series are replaced by uniform continuity. Apart from this, only elementary arguments are needed. In particular, neither integration nor the rule for derivatives of inverse functions are used except for (R5) and (R4).

The moral of the story is that actually any of the roles listed above can be chosen as starting points without inherently being on foundationally shaky ground. None is really

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[1] In this paper, "exponential function" refers to a constant (always positive) base and varying exponent, as opposed to "power function" where the exponent is fixed and the base varies. The word "natural" is added in front of "exponential" or "logarithm" when the base is $e$.

more fundamental than the others. The choice of approach becomes a paedagogical one, with an instructor's (or individual student's) choice as to which steps to cover in what level of detail being on the same level as a reader's choice as to which lemmas to skip over while reading a research paper.

## 1. Rational exponentials and logarithms

Let's start by explicitly stating the fundamental properties of rational exponentials and their inverses, rational logarithms. It is worth temporarily introducing awkward-looking "named function" notation for exponentials, in part to make it easier to "forget" certain familiar algebraic properties that we don't want to use until after Theorem 5, below.

**Proposition 1** (Rational exponentials)**.** *For a fixed $b > 0$, the formula $E_b(p/q) = b^{p/q} = \sqrt[q]{b^p}$ defines a function $E_b : \mathbf{Q} \to \mathbf{R}^+$ satisfying the* fundamental relation $E_b(x+y) = E_b(x)E_b(y)$. *The function $E_b$ is called the* rational exponential with base $b$*, and $E_b(1) = b$.*

**Proposition 2** (Rational logarithms)**.** *For a fixed $b > 0$, $b \neq 1$, the formula $L_b(x) = \log_b x = E_b^{-1}(x)$ defines a function $L_b : \mathrm{Range}(E_b) \to \mathbf{Q}$ satisfying the* fundamental relation $L_b(xy) = L_b(x) + L_b(y)$. *The function $L_b$ is called the* logarithm with base $b$ *and $L_b(b) = 1$.*

The fundamental relations are enough to recover the other standard rules for exponentials and logarithms. For instance, $E_b(0) = 1$ since $E_b(x+0) = E_b(x)E_b(0)$; and $E(xy) = E(x)^y$, $(y = p/q)$, first by induction on $p$ when $q = 1$, and then by raising both sides to the $q$th power. It also follows that $E_b$ is monotonic (unless $b = 1$) and thus is indeed invertible.

## 2. Irrational exponents and continuous extensions

A sketch-graph of $E_b : \mathbf{Q} \to \mathbf{R}^+$ (for fixed $b$) looks like an infinitely fine mesh of points coalescing into a line, i.e. $E_b$ appears to be continuous as a function of $\mathbf{Q}$. This suggests defining $b^x$ for all real $x$ by just "connecting the dots". In fact, students (and many post-students) implicitly believe the following general principle, and can often state it themselves with a bit of terminological assistance:

**Theorem 3** (FALSE!)**.** *Suppose $Q$ is a dense subset of a set $R$, and let $f : Q \to \mathbf{R}$ be a continuous function. Then there is a unique continuous function $\tilde{f} : R \to \mathbf{R}$ which extends $f$, i.e. $\tilde{f}$ restricted to $Q$ is the same as $f$.*

To find $\tilde{f}(x)$, we ought to be able to just approximate $x$ by elements $x' \in Q$ and set $\tilde{f}(x) = \lim_{x' \to x} f(x')$. The continuity of $f$ should somehow imply that this limit exists and is unique. However, consider any function $g : \mathbf{R} \to \mathbf{R}$ which is continuous except for an essential discontinuity at an irrational number $\xi$, and let $f = g|\mathbf{Q}$. The function $f$ is continuous, even if $g$ is not, since the discontinuity at $\xi$ is invisible from inside $\mathbf{Q}$. Theorem 3 clearly breaks down trying to define $\tilde{f}(\xi)$.

**Theorem 4** (Continuous extension from a dense set)**.** *Theorem 3 becomes true if $f$ is uniformly continuous on all bounded subsets of $Q$. In this case, $\tilde{f}$ is uniformly continuous on all bounded subsets of* $\mathbf{R}$.

The proof starts along the lines outlined above, but involves the completeness of the codomain $\mathbf{R}$ as well as uniform continuity (see [4, Theorem 15.4], for instance.) It is of course unlikely that one would want to cover this in full detail in a beginning calculus course. However, it can certainly be discussed at a comparable level of rigor as, for instance, the Intermediate Value Theorem. The experience of having believed Theorem 3 also makes students treat other results involving continuity and the nuts and bolts of the structure of the real number system with more respect.

In fact, I claim that this theorem *should* be mentioned on some level in any "serious" calculus course, because it underlies the use of technology in mathematics! Plotting a function on a computer or graphing calculator involves the machine calculating values on a fairly fine mesh of points and interpolating in between on the screen. We then further interpolate between the individual pixels with our eyes. The uselessness of technology for graphing the Dirichlet function ($D(x) = 1$ if $x \in \mathbf{Q}$ and $D(x) = 0$ if $x \in \mathbf{R} - \mathbf{Q}$), or even the function $S(x) = \sin(1/x)$ too close to 0, arises exactly from some sort of lack of continuity and the associated difficulties in approximation. To what level it is appropriate to discuss the uniformity hypothesis varies with the course and the students, of course, but the function $S(x)$ begs at least a mention of it.

We verify below that $E_b : \mathbf{Q} \to \mathbf{R}^+$ is uniformly continuous on bounded intervals. For now, assume this and define $b^x$ for $x \in \mathbf{R}$ as $\tilde{E}_b : \mathbf{R} \to \mathbf{R}^+$, given by continuous extension. The fundamental relation extends to $\mathbf{R}$ by continuity, and shows that the extended function is also monotonic (unless $b = 1$) and unbounded, so its range is all of $\mathbf{R}^+$ and it has a continuous and monotonic inverse $L_b(x) = \log_b x$.

**Theorem 5** (Exponentials and logarithms)**.** *There is a one-to-one correspondence between*

1. *Continuous functions $E : \mathbf{R} \to \mathbf{R}^+$, satisfying the fundamental relation $E(x+y) = E(x)E(y)$, called exponentials.*
2. *Nonconstant continuous functions $L : \mathbf{R}^+ \to \mathbf{R}$, satisfying the fundamental relation $L(xy) = L(x) + L(y)$, called logarithms; and*
3. *Positive real numbers $b$, called bases.*

*The corresponding functions $E$ and $L$ are inverses; $E(1) = b$, $L(b) = 1$, $E(x) = b^x$, and $b^{L(x)} = x$.*

*Sketch-proof.* If $E(x)$ is continuous and satisfies the fundamental relation (extended from $\mathbf{Q}$ to $\mathbf{R}$ as above), then $E(px/q) = E(x)^{p/q}$ as discussed after Proposition 1. This persists for $p/q$ replaced by any real number by continuity, and so we conclude that $E(x) = E(1)^x$. The results for logarithms follow by similar arguments and inversion. □

We now dispense with the cumbersome $E_b$ and $L_b$ notation introduced at the beginning, but continue to remember that $b$ is fixed and $x$ is what varies.

## 3. Uniform continuity of $b^x$

To show $b^x$ is a uniformly continuous function of $x$ on bounded subsets of $\mathbf{Q}$, and thus to complete the extension to $x \in \mathbf{R}$, we use the formula

$$|b^x - b^y| = b^y |b^{x-y} - b^0|.$$

Since $b^x$ is monotonic, it is bounded on any bounded interval by its values on the end-points, and so all we need to prove is the following

**Proposition 6.** $\lim_{x\to 0} b^x = 1$ *($x \to 0$ through* $\mathbf{Q}$*).*

We prove the case where $b > 1$ and $x \to 0^+$. The other cases follow similarly or by substituting $1/b$ for $b$.

*First proof.* By the Pinching Theorem, it suffices to show $1 \le b^x \le 1 + xb$ for $0 < x < 1$. Fix $x$ and consider the function $g(b) = 1 + bx - b^x$. Since $b$ is now the variable, the $b^x$ term is a power function (with constant rational exponent), not an exponential. In fact, $g'(b) = x(1 - b^{x-1}) > 0$, so $g(b)$ is increasing. Since $g(1) = x > 0$, this means $g(b) > 0$ for all $b \ge 1$. $\square$

The above proof uses the fact that a function whose derivative is positive on an interval is increasing there, a consequence of the Mean Value Theorem. With a bit more effort, we can avoid this.

**Lemma 7.** *If $b > 1$ and $n$ is a positive integer, then $1 \le b^{1/n} \le 1 + b/n$.*

*Proof.* This first inequality is clear. For the second, assume $b^{1/n} > 1 + b/n$. Then $b > (1 + b/n)^n > 1 + nb/n = b + 1$, a contradiction. $\square$

*Second proof of Proposition 6.* By Lemma 7 and the Pinching Theorem, $b^{1/n} \to 1$. So the limit is 1 as $x \to 0^+$, since the values of $b^x$ evaluated for $x$ in between the points $\{1/n\}$ are constrained by monotonicity. $\square$

We can even make the last sentence more explicit: find the integer $n$ such that $1/(n+1) \le x < 1/n$. Then $n + 1 \ge 1/x$, and so $n \ge 1/x - 1 = (1 - x)/x > 0$. Hence, using Lemma 7 and monotonicity, we obtain an alternate pinching inequality

$$1 \le b^x \le b^{1/n} \le 1 + b/n \le 1 + b\frac{x}{1-x} \tag{1}$$

Here is yet another proof, a sneaky one using geometric series (with a nod to [8]). By Lemma 7, $1 \le b^{1/q} \le 1 + b/q$, so that $1 \le b^{p/q} \le (1 + b/q)^p$. Expand the final expression using the Binomial Theorem, noting that $\binom{p}{k} \le p^k$, so that $\binom{p}{k}(b/q)^k \le b^k(p/q)^k \le b(p/q)^k$. Thus we get a finite subseries of the infinite geometric series with first term $b(p/q)$ and ratio $p/q$. Summing this series recovers the equation (1).

## 4. The derivative of $b^x$

**Theorem 8.** *There is function $\lambda : \mathbf{R}^+ \to \mathbf{R}$ such that for all $b > 0$, $\frac{d}{dx} b^x = \lambda(b) b^x$. In particular, $b^x$ is a differentiable function for each $b$.*

We postpone using the name "ln" for $\lambda$, until we show that it is indeed a logarithm in the sense of Theorem 5.

**Lemma 9.** *Suppose $\alpha > 0$ and $z > 1$. Then $(1 + \alpha)^z \ge 1 + \alpha z$.*

*First Proof (requiring MVT).* We proceed as in the first proof of Proposition 6. Let $g(\alpha) = (1 + \alpha)^z - (1 + \alpha z)$. We have $g'(\alpha) = z(1 + \alpha)^{z-1} - z > 0$ and so $g$ is increasing (MVT!) for $\alpha > 0$. Since $g(0) = 0$, this implies the Lemma. By continuity, we may restrict to $z \in \mathbf{Q}$. $\square$

*Second Proof (no MVT but messier).* Suppose $z \in \mathbf{Q}$, so $z = p/q$, $p > q$. The Lemma is equivalent to the relation

$$(1+\alpha)^p \geq (1+\alpha p/q)^q \tag{2}$$

Expand both sides using the Binomial Theorem, and let $L_k$ and $R_k$ be the coefficients of $\alpha^k$ on the left hand and right hand sides respectively. Then

$$L_k = \binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!} \tag{3}$$

$$R_k = \binom{q}{k}(p/q)^k = \frac{q(q-1)\dots(q-k+1)}{k!}\left(\frac{p}{q}\right)^k. \tag{4}$$

However, since $p > q$, $(p/q)(q-i) \leq (p-i)$ for all $0 \leq i < q$ and so $L_k \geq R_k$ for $0 \leq k \leq p < q$. Also, $L_k > 0 = R_k$ for $p < k \leq q$. Since $\alpha > 0$, this implies inequality (2). □

*Proof of Theorem 8.* The difference quotient for computing $E_b'(x)$ is

$$\frac{b^{x+h} - b^x}{h} = \frac{b^h - 1}{h} b^x$$

so it suffices to show that the function $F(x,h) = (x^h - 1)/h$, defined for $h \neq 0$, tends to a limit as $h \to 0$. Graphing $F(x,h)$ for various $h$ strongly suggests that this is the case, and that we should be able to prove it by some sort of pinching. An obvious idea is to calculate $F(x,h) - F(x,k)$, for small positive $h$ and small negative $k$, but this is a mess. Instead, we remark that

$$F(x,-h) = \frac{x^{-h}-1}{-h} = x^{-h}\frac{1-x^h}{-h} = x^{-h}F(x,h), \qquad \text{and} \tag{5}$$

$$F(x,kh) = \frac{x^{kh}-1}{kh} = \frac{1}{k}F(x^k,h). \tag{6}$$

Suppose now that $h > 1$ and $x > 1$. Lemma 9 implies that

$$F(x,h) = \frac{(1+(x-1))^h - 1}{h} \geq \frac{1+h(x-1)-1}{h} = x-1. \tag{7}$$

Applying this to the right hand side of (6) (with $k > 0$), we obtain

$$F(x,kh) \geq (1/k)(x^k - 1) = F(x,k) \tag{8}$$

which implies $F(x,k)$ increases as a function of $k$. Chasing through the sign changes and applying (5) as required, we discover that this remains true for $0 < x < 1$ and regardless of the sign of $k$.

Finally, letting $h \to 0$ in (5), we see $F(x,h)/F(x,-h) \to 1$. Since $F(x,h)$ increases as a function of $h$, this implies $F(x,h)$ is pinched to a limit $\lambda(x)$ as $h \to 0$. □

## 5. The antiderivative of $x^{-1}$

Observe that $F(x,h) = \int_1^x t^{h-1}\,dt$. If we set $h = 0$, $F(x,h)$ is no longer defined, but $\int_1^x t^{-1}\,dt$ is still some function of $x$ by the Fundamental Theorem of Calculus.

**Theorem 10.** *$\int_1^x t^{-1}\,dt = \lambda(x)$ and so $\frac{d}{dx}\lambda(x) = 1/x$.*

*Proof.* The obvious idea is to just write

$$\int_1^x t^{-1}\,dt = \int_1^x \left(\lim_{h\to 0} t^{h-1}\right)dt = \lim_{h\to 0}\int_1^x t^{h-1}\,dt = \lim_{h\to 0} F(x,h) = \lambda(x).$$

This involves interchanging the limit and integral operations, which requires uniform convergence. Alternatively, let $\Lambda(x) = \int_1^x t^{-1}\,dt$. We claim $\Lambda(x) = \lambda(x)$. Suppose $x > 1$. If $h > 0$, then $x^{-1+h} < x^{-1} < x^{-1+h}$ so $F(x,-h) < \Lambda(x) < F(x,h)$ after integration. But $\lambda(x)$ is the only function which satisfies this as $h \to 0$. □

## 6. $\lambda(x)$ IS A LOGARITHM

We now have three different but compatible possible definitions of $\lambda(x)$. The first two are via the roles (R3) and (R5). The third is the "limit" definition involving the ratio $F(x,h)$, which which actually underlies both of the others. We can use any of these to prove the following

**Proposition 11.** *The function $\lambda(x)$ has the following properties:*

1. $\lambda(1) = 0$, $\lambda(x) > 0$ *for* $x > 1$, *and* $\lambda(x) < 0$ *for* $x < 1$.
2. $\lambda(xy) = \lambda(x) + \lambda(y)$.
3. $\lambda$ *is an unbounded increasing continuous function.*

*It is thus is a logarithm.*

It suffices to prove properties 1 and 2, and the continuity part of property 3. The increasing and unbounded part of 3 is then automatic, since for $y > 1$ we get $\lambda(xy) = \lambda(x) + \lambda(y) > \lambda(x)$ and $\lambda(x^n) = n\lambda(x)$, and $\lambda$ is not identically 1.

*"Limit" proof.* For property 1, suppose $x > 1$. If $h > 0$, then $x^h > 1$, so $hF(x,h) = x^h - 1 > 0$ and thus $F(x,h) > 0$. If $h < 0$, then $x^h < 1$, so $hF(x,h) < 0$ and $F(x,h) > 0$ as before. Thus the limit function $\lambda(x)$ is positive. The case $x < 1$ is similar.

For property 2, let $h \to 0$ in the following identity:

$$F(xy,h) = \frac{x^h(y^h-1) + x^h - 1}{h} = x^h F(y,h) + F(x,h). \tag{9}$$

Finally, to see $\lambda$ is continuous, it suffices to check that $\lambda(xy) - \lambda(x) = \lambda(y)$ can be made arbitrarily small for $y$ close to 1. But this follows from fixing some small $h > 0$ in the pinching inequality $F(y,-h) \le \lambda(y) \le F(y,h)$. □

*"Derivative" proof.* Property 1 is immediate. For property 2, calculate $\frac{d}{dz}\big|_{z=0}(xy)^z$ in two ways. On the one hand, it equals $\lambda(xy)(xy)^z\big|_{z=0} = \lambda(xy)$. On the other hand, writing $(xy)^z = x^z y^z$ and using the product rule, it equals $\lambda(x) + \lambda(y)$. To prove continuity, use the limit definition of the derivative and proceed as in the previous proof. □

*"Integral" proof.* Property 1 and continuity follow directly from basic properties of the integral and the Fundamental Theorem of Calculus. Property 2 follows from the standard calculation $\frac{d}{dx}\lambda(xy) = \lambda'(xy)y = y/(xy) = 1/x$. Since $\lambda(x)$ is itself an antiderivative of $1/x$, $\lambda(xy) - \lambda(x)$ is independent of $x$, i.e. $\lambda(xy) = \lambda(x) + f(y)$ for some function $f(y)$. Finally $\lambda(1y) = \lambda(1) + f(y) = f(y)$, so $f(y) = \lambda(y)$. □

Since $\ln x = \lambda(x)$ is a logarithm, there is necessarily some unique base $e$ such that $\ln e = 1$, and role (R4) follows by Theorem 5. Indeed, all logarithms and exponentials are thus not only continuous but also differentiable.

Some traditional calculus texts, such as [5, 6], prepare the shock of the "integral" definition of ln by first defining logarithms as as *differentiable* functions satisfying the fundamental relation $L(xy) = L(x) + L(y)$. Then it is shown by change of variable that $L'(x) = L'(1)/x$ and so ln is "natural" in that $L'(1) = 1$. Differentiability as a requirement, however, is somewhat unnatural in what is otherwise so far a calculus-free concept.

## 7. A PARALLEL WITH $\sin x$, AND THE NATURALITY OF $e$

The above approach to $\frac{d}{dx}b^x$ closely parallels the standard proof that $\frac{d}{dx}\sin x = \cos x$. Trigonometric identities reduce this to computing the limits

$$\lim_{h\to 0}\frac{\sin h}{h} \quad \text{and} \quad \lim_{h\to 0}\frac{1-\cos h}{h}.$$

This is analogous to our use of the limit $\lim_{h\to 0} F(b,h)$. Unfortunately, we no longer have the nice geometric picture as in the $(\sin h)/h$ case, and so we have to work much harder to show the limit actually exists. It is of course a function of $b$.

The base $e$ is "natural" in that $F(e,h) \to 1$, as opposed to some other constant, just as radians are the "natural" angle measure for which $(\sin h)/h \to 1$. Thus use of base $e$ and radian measure eliminates annoying multiplicative constants in calculations, to the extent we generally don't bother remembering the formulas for $\frac{d}{dx}b^x$ for general $b$ or for $\frac{d}{dx}\sin$ in degrees. When necessary, either can always be obtained via chain rule from the "natural" form.

Indeed, once we "know" the derivative of $e^x$ and $\sin x$, we can even forget the limits $(\sin h)/h$ and $F(b,h)$, since we can recalculate them using l'Hospital's rule. In fact, it was the short note [1], placing $\ln b$ among the functions $F(b,h)$ via l'Hospital's rule, that planted the seed for the present paper in my head while I was an undergraduate.

There are a number of ways of obtaining the limit $e^x = \lim_{n\to\infty}(1+x/n)^n$ in (R7), as well as the related limit $e = \lim_{h\to 0}(1+h)^{1/h}$, as a consequence of the other roles. To complete my claim that any of the roles could in fact be chosen as starting points in the exponential and logarithm "story", it only remains to show by bare hands that $s_n = (1+x/n)^n$ converges to a limit. To see this, expand $s_n$ by the Binomial Theorem. As $n \to \infty$, the coefficients of each $x^k$ approach those in the power series expansion of $e^x$ from below, and the arguments used in [8] to prove the convergence of the power series solution to (R6) also work for (R7).

## 8. CLASSROOM USE

Clearly, the number of details in a fully rigorous presentation along the lines of this paper is excessive for classroom use. (This is an indisputable advantage of the "traditional approach" starting with (R5)—it is short enough, even in all its gory detail, not to exhaust the instructor's patience. The danger is that the instructor may thus not notice whether it exhausts the *students'* patience.) What can be left out, and at what places can do various parts fit into the standard calculus curriculum?

The argument actually falls naturally into several modular segments, each of which can be mentioned only briefly, worked through in detail, or even assigned as a sequence of guided exercises:

- (S1) (a) Properties of $b^x$ for $x \in \mathbf{Q}$
  - (b) (Uniform) continuity of $b^x$ for $x \in \mathbf{Q}$
  - (c) Extension to $x \in \mathbf{R}$
- (S2) (a) Reduction of $\frac{d}{dx}b^x$ to $F(x,h)$
  - (b) $F(x,h)$ increases with $h$
  - (c) $F(x,h)$ is pinched to a limit.
- (S3) $\ln x$ as antiderivative
- (S4) $\ln x$ as logarithm
- (S5) The role of $e$

Segment (S1) can be covered any time after continuity, though the proofs are easier if derivatives, MVT, and increasing functions have been covered. Because of the parallel with $\frac{d}{dx}\sin x$, (S2) should either follow that development, or directly follow (S1) if that has been delayed post-MVT. All this (and also (S4), if desired) can be done prior to introducing integration. There is no reason (S5) has to be last.

In my moderately theoretical first-year calculus course based on [5], I emphasize the "mistake" (Theorem 3) in (S1), but wave my hands though the proof of the corrected Theorem 4. I don't prove (S2)(b) (which students find very believable) but do the $F(x,h)/F(x,-h)$ calculation in (S2)(c). I've placed (S1),(S2), (S4) both before and after covering the Mean Value Theorem and before and after integration, the choice governed by how far I am in the course by Christmas, and how comfortable the class seems with bypassing the textbook's arguments.

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