EPIMORPHIC SUBGROUPS OF SL_n, TREES, AND THE TITS BOUNDARY

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ABSTRACT. A geometric construction is presented to visualize the structure of solvable algebraic epimorphic subgroups H of $SL_n(\mathbf{C})$ or $SL_n(\mathbf{R})$ which are normalized by a maximal torus T. In particular, we classify all such minimal subgroups up to conjugacy. The unipotent radicals of such H correspond to trees on n vertices, and their maximal subtori to points in a subset of the Tits boundary $\mathcal{B}$ of T, determined by a natural embedding of the tree in $\mathcal{B}$.

Examples are given for $n = 2, \ldots, 6$. The same construction applies to other semisimple groups, with the tree replaced by a more complicated combinatorial object.

A subgroup H of an algebraic group G over some field is called *epimorphic* (denoted $H <_e G$) if in any (finite-dimensional) representation $G \to GL(V)$ any vector v fixed by H is fixed by all of G. The concept was introduced by Bergman in 1970, and has recently been studied, along with the "opposite" property of observability, by Bien and Borel in [BB1, BB2, BB3] and by Grosshans [Gro]. There are various equivalent representation-theoretic conditions.

Recently, epimorphic subgroups have proved useful in the study of group actions on (real) manifolds and homogeneous spaces preserving geometric structures, e.g., [Per] and [Moz, Wei].

We consider the base field $\mathbb{F}$ to be either either $\mathbb{C}$ or $\mathbb{R}$. The following properties of epimorphic subgroups are well known:

1. If $H < H' < G$ and $H <_e G$, then $H' <_e G$.
2. If $H <_e G$, then a solvable subgroup $H' < H$ is epimorphic in G. By a result of Bien and Borel, if $\mathbb{F} = \mathbb{R}$ and G is generated by $\mathbb{R}$-unipotents (for instance, G is semisimple with no compact factors), then H' is split.
3. $H <_e G$ if and only if the Zariski closure of H is epimorphic in G.

Accordingly, it is important to understand the structure of minimal algebraic epimorphic subgroups, which are necessarily solvable.

Necessary and sufficient root-space conditions for epimorphicity are known when a solvable algebraic subgroup contains, or is at least normalized by, a maximal torus T. In this paper we restate these conditions in terms of graphs (trees) and the Tits boundary of T, in a way which lets us classify all examples, especially the minimal ones.

The structure of non-T-normalized solvable algebraic epimorphic subgroups (i.e. of the form $H = SU$ with S a torus normalizing a unipotent group U which

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is *not* normalized by any maximal torus $T > S$) is more complicated and poorly understood. However, many and possibly all (see Section 5.2) non-T-normalized solvable epimorphic subgroups $H <_e G$ are constructed from maximal-torus-normalized epimorphic subgroups of subgroups of G, so understanding the structure in the T-normalized case is of interest in the more general setting.

1. STATEMENT OF RESULTS

Let $G = SL_n(\mathbb{F})$ and T be a maximal torus.

Theorem 1.1 (Main result). *Let $H = SU$ represent a solvable algebraic subgroup normalized by T, where S is a subtorus of T and U is the unipotent radical. There is a natural one-to-one correspondence between*

1. *U such that $H <_e G$ is minimal, up to conjugation by the Weyl group, and*
2. *trees τ on n vertices, up to isomorphism.*

There is a further one-to-one correspondence between

1. *S (necessarily one-dimensional) such that $H <_e G$ is minimal, and*
2. *points in a convex subset of the Tits boundary $\mathcal{B}$ of T determined by the natural embedding of the tree τ in $\mathcal{B}$. (See Section 4.1 for details and Figure 3 for an example.)*

Corollary 1.2. *For $n \geq 4$, there are several distinct classes of minimal algebraic epimorphic subgroups of SL_n with nonconjugate unipotent radicals, and the number of such classes grows very rapidly with n.*

This extends results in [BB1, BB3], where the example corresponding to the straight-line tree is constructed algebraically (for any semisimple G, not just SL_n).

The unipotent part of Theorem 1.1 specializes from the following, where $B > T$ is a fixed Borel subgroup:

Theorem 1.3. *There is a natural one-to-one correspondence between*

1. *minimal solvable algebraic subgroups $H <_e G$ such that $T < H < B$.*
2. *trees on n vertices, with the vertices numbered subject to the condition $(\dagger)$ that no sequence of 3 adjacent vertices is increasing or decreasing, (i.e., the path fragment $1 — 3 — 4$ is not allowed, but $1 — 4 — 3 — 5$ is).*

Theorem 1.4. *There is a natural one-to-one correspondence between*

1. *solvable algebraic subgroups $H <_e G$ such that $T < H < B$ (H not necessarily minimal).*
2. *connected graphs on n numbered vertices[1], subject to the condition $(\ddagger)$ that whenever vertices v and w are joined by a path along which the vertex numbers are all increasing (or decreasing), then v and w are joined by an edge.*

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[1] Our graphs have non-directed edges between distinct vertices and two vertices are joined by at most one edge.

The unipotent radical U of H can be read off the vertex numbering, with an edge vw (with $v < w$ in the ordering) corresponding to the elementary subgroup $\mathbb{F}e_{vw} < H$ when B is taken to be upper triangular. The effect of a Weyl conjugation σ is to permute the numbers of the vertices, with $\sigma(H) < B$ provided the numbering constraints on the graph remain satisfied.

The Tits boundary construction in the second part of Theorem 1.1 also generalizes to nonminimal H (Section 4.2). The size of the convex subset in $\mathcal{B}$ measures the amount of flexibility in choosing $S < T$ so that SU is epimorphic.

For the rest of this paper, we omit the word “algebraic” in “algebraic subgroup”, though it remains always understood. The applicability of our results to more general semisimple G, and to non-T-normalized H, is discussed in Section 5.

2. THE CASE $T < H < B$

2.1. **Proof of Theorems 1.3 and 1.4.** It is convenient to use the Lie algebras $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g} = \mathfrak{sl}_n$ corresponding to $T < H < G$, etc. Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}$ be the root space decomposition with $\mathfrak{g}_{ij} = \mathbb{F}e_{ij}$. Let $\mathfrak{n} = \bigoplus_{i<j} \mathfrak{g}_{ij}$, the nilpotent part of the standard Borel subalgebra $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$.

Lemma 2.1. *Let $\mathfrak{h}' \subset \mathfrak{n}$ be an $(\operatorname{ad}\mathfrak{t})$-invariant subspace. Then $\mathfrak{h}'$ is the direct sum of some of the $\mathfrak{g}_{ij}$ ($i < j$).*

Proof. If $\mathfrak{h}'$ contains any element x which has a nontrivial component in some $\mathfrak{g}_{ij}$, then $(\operatorname{ad}\mathfrak{t})x$ contains $\mathfrak{g}_{ij}$, i.e., all the other components can be annihilated and the whole span $\mathbb{F}e_{ij}$ obtained. Containment in $\mathfrak{n}$ implies $i < j$. $\square$

This allows us to define a mapping

$$\mathcal{G} : \{(\operatorname{ad}\mathfrak{t})\text{-invariant subspaces of } \mathfrak{b}\} \longrightarrow \{\text{graphs on } n \text{ numbered vertices}\}$$

as follows: in the graph $\mathcal{G}(\mathfrak{h})$, the vertices $k < l$ are joined by an edge iff $\mathfrak{g}_{kl} \subset \mathfrak{h}$. The mapping is clearly onto, and the fibers are isomorphic to the set of subspaces of $\mathfrak{t}$. We now restrict to the case $\mathfrak{t} \subset \mathfrak{h}$ so that $\mathcal{G}$ is one-to-one.

Lemma 2.2. *$\mathfrak{h}$ is a subalgebra iff $\mathcal{G}(\mathfrak{h})$ satisfies condition $(\ddagger)$ of Theorem 1.4.*

Proof. Suppose $i < j$, $k < l$, and $i \leq k$. Then $[\mathfrak{g}_{ij}, \mathfrak{g}_{kl}] = \mathfrak{g}_{il}$ if $j = k$, else 0. Thus $\mathfrak{h}$ is a subalgebra iff for any “Lie bracket string” $[\ldots[[\mathfrak{g}_{ab}, \mathfrak{g}_{bc}], \mathfrak{g}_{cd}]\ldots, \mathfrak{g}_{yz}]$, with $a < b < \cdots < z$, we have $\mathfrak{g}_{az} \subset \mathfrak{h}$. The precise placement of the inside brackets (i.e. the order of applying brackets within the string) does not matter, since one of the terms in the Jacobi identity always vanishes.

The existence of a path along which the vertex numbers all increase (or all decrease) is equivalent to the existence of a string as described, and the existence of an edge between the endpoints of this path is equivalent to the iterated Lie bracket of the string being in $\mathfrak{h}$. $\square$

Proposition 2.3 (Pommerening [Pom], [BB1, BB3], [Gro, Lemma 3.11])**.** *The epimorphic solvable subgroups H of a semisimple group G, with $T < H$, are of the form $H = TU_\Psi$, where $U_\Psi = \langle U_\alpha \,|\, \alpha \in \Psi \rangle$. Here Ψ is a subset of the positive roots with respect to T, such that $\langle T, U_\alpha, U_{-\alpha} \,|\, \alpha \in \Psi \rangle = G$.*

Lemma 2.4. *The subalgebra $\mathfrak{h}$ is epimorphic in $\mathfrak{g}$ iff $\mathcal{G}(\mathfrak{h})$ is connected.*

Proof. It is necessary and sufficient that $\mathfrak{g}_{ij}$ for any $i \neq j$ be represented as a Lie bracket string of various $\mathfrak{g}_{kl}$ or $\mathfrak{g}_{lk}$ where $\mathfrak{g}_{kl} \subset \mathfrak{h}$ (and hence $k < l$). This is equivalent to i and j being joined by a path in $\mathcal{G}(\mathfrak{h})$. □

This proves Theorem 1.4. We henceforth write interchangably $\mathcal{G}(\mathfrak{h}) = \mathcal{G}(H) = \mathcal{G}(\Psi)$ when $H = TU_\Psi$.

To complete the proof of Theorem 1.3 it suffices to prove the following

Lemma 2.5. *An epimorphic subalgebra $\mathfrak{h}$ is minimal (i.e., contains no proper subalgebra which is epimorphic in $\mathfrak{g}$ and still contains $\mathfrak{t}$) iff $\mathcal{G}(\mathfrak{h})$ is a tree satisfying condition (†).*

Proof. It is clear that minimality implies that the graph be a tree, in which case condition (‡) becomes condition (†). Conversely, a tree with condition (†) satisfies condition (‡) and no more edges can be removed without disconnecting the graph. □

2.2. **Weyl conjugation.** The effect of a Weyl conjugation σ is to rearrange the order of the rows and columns of $\mathfrak{sl}_n$ (and $\mathfrak{h}$), which corresponds to renumbering the vertices of $\mathcal{G}(\mathfrak{h})$. In general, Weyl conjugation does not preserve $\mathfrak{b}$, and in this case the correspondence given by $\mathcal{G}$ breaks down, since $\sigma(\mathfrak{h})$ will contain some $\mathfrak{g}_{ij}$ with $i > j$, while $\mathcal{G}^{-1}(\sigma(\mathcal{G}(\mathfrak{h})))$ will contain $\mathfrak{g}_{ji}$. However, provided $\sigma(\mathfrak{h})$ remains in $\mathfrak{b}$, this does not occur. Thus we can classify the epimorphic $\mathfrak{h} \subset \mathfrak{b}$ containing $\mathfrak{t}$ by just "forgetting the vertex numbering". Hence to prove Theorem 1.1, in the case that $T < H$, is suffices to prove the following

Lemma 2.6. *Any tree can be numbered in accordance with Condition* (†).

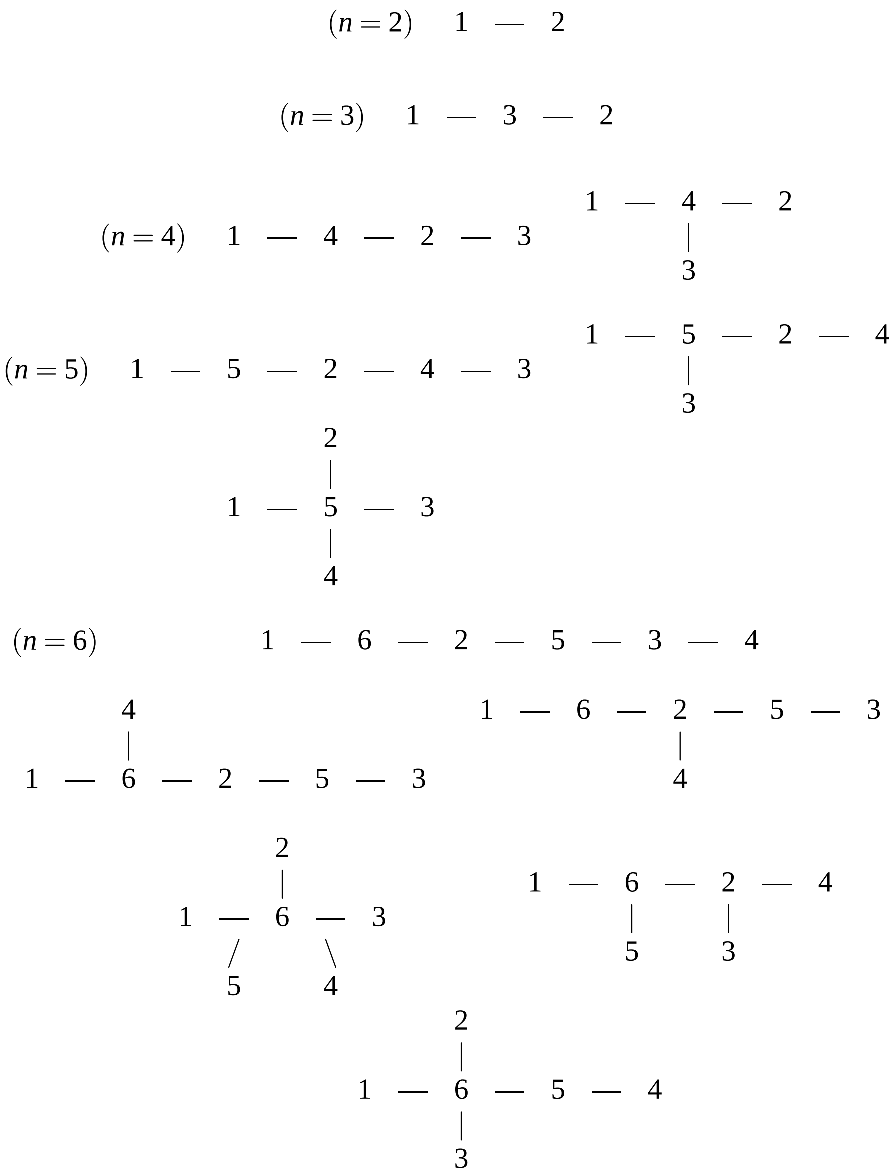
Proof. We prove any tree with n vertices can be numbered from k to $(n+k-1)$ for any k, by induction on n. It is clear for $n = 1$ and $n = 2$. Now take a larger tree and choose any vertex. Label it k. Label the vertices joined to k as $(n+k-1)$, $(n+k-2)$, $\ldots$. Now remove the numbered vertices, leaving one or more trees with fewer vertices. By induction, label these starting at $(k+1)$.

Consider the paths we just created. They are of two forms. The first is $k - s - t - r$, with $t - r$ being given by the induction step (r, s, t are placeholders). Now $k < s$, $t < r$, and $t < s$ by construction, satisfying the numbering condition. The second is $r' - t' - s - t - r$ (k is adjacent to s, but is not in this path) with $t - r$ and $t' - r'$ given by the inductive step. Again, $t < r$, $t' < r$, $t < s$, and $t' < s$, so Condition (†) is satisfied and the numbering is complete. □

2.3. **Counting the types; examples.** The number of conjugacy types of minimal U, and hence of minimal epimorphic $T < H < B$, is the the number C_n of isomorphism types of trees on n vertices. This can be computed (nontrivially) via generating functions, for instance, [HP, section 3.2 and Table A7]). $C_n > 1$ for $n \geq 3$ and increases very rapidly with n, for instance $C_{10} = 106$ and $C_{20} = 823,065$.

The possible tree types for $n = 2, \ldots, 6$ (corresponding to minimal solvable epimorphic subgroups of SL_2 through SL_6 containing T), with the vertices numbered in one possible way to satisfy Condition (†), are shown in Figure 1.

FIGURE 1. Tree types for SL_2 through SL_6 (all tree types, with one numbering satisfying Condition (†) for each)



In particular, in any SL_n we have the epimorphic subgroup

$$H = \begin{pmatrix} * & & & * \\ & * & 0 & * \\ & & \ddots & \vdots \\ & & & * \end{pmatrix}$$

corresponding to the hub-and-spokes tree numbered as in the Figure.

Bien and Borel construct the following example[2] ([BB1][BB3, Prop. 2.9]), in any semisimple group G. Choose a set of simple roots Δ and partition it into 2 parts I and J so that elements of I (respectively J) are mutually orthogonal. Then let H include T and the root spaces in $I \cup -J$. It turns out that this is a subgroup and is epimorphic, though it does not lie in the Borel subgroup determined by Δ being positive. In SL_n, this construction corresponds to taking an off-diagonal and flipping each second entry across the diagonal. It is clear (since $\mathcal{G}$ "forgets" the flipping) that after a suitable Weyl conjugation to end up upper triangular, this example corresponds to the straight-line tree.

3. THE T-NORMALIZED CASE AND SUBTORI

Now we relax the condition that $\mathfrak{t} \subset \mathfrak{h}$ to $\mathfrak{h}$ only being $(\operatorname{ad}\mathfrak{t})$-invariant, i.e. H is normalized by T. Proposition 2.3 generalizes to

Proposition 3.1 (Pommerening [Pom], Bien-Borel [BB3])**.** *The epimorphic solvable subgroups H of a semisimple group G, such that H is normalized by T, are of the form $H = SU_\Psi$, where U_Ψ is as in Proposition 2.3, and $S < T$ satisfies the following condition: the only $\lambda \in X(T)$ such that $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Psi$ and such that $\lambda|S = 0$, is $\lambda = 0$.*

In particular, this means that the graph construction $\mathcal{G}$ in Section 2.1 is meaningful and depends only on Ψ.

3.1. Identifying minimal tori.

The following proposition reformulates Proposition 3.1 and generalizes the subtorus construction in example [BB3, Prop. 2.9].

Proposition 3.2. *The minimal epimorphic solvable subgroups H of a semisimple group G, such that H is normalized by T, are of the form $H = SU_\Psi$, where*

1. *U_Ψ is as in Proposition 2.3 and minimal, and*
2. *S is any one-dimensional torus intersecting the subset*

$$\mathcal{C}(\Psi) = \{s \in T \mid \alpha(s) > 0 \text{ for all } \alpha \in \Psi\}.$$

One possible S is the "diagonal" torus defined by the relations $\alpha|S = \beta|S$ for all $\alpha, \beta \in \Psi$, or $S = \bigcap_\Psi \ker(\alpha - \beta)$. In general, S may be regular or singular.

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[2] Actually, their example is for a minimal T-normalized group, but for the moment we treat only the unipotent radical.

Proof. Since $G = \langle T, U_\alpha, U_{-\alpha}\rangle_{\alpha\in\Psi}$, we have $\mathbb{Z}[\Psi] = \Phi(T,G)$, the root lattice, and $\mathbb{R}[\Psi] = \Phi(T,G)\otimes\mathbb{R}$. Since H is minimal, Ψ must be minimal, and hence Ψ forms a basis for $\Phi(T,G)\otimes\mathbb{R}$. Any $\lambda \in X(T)$ such that $(\lambda,\alpha) \geq 0$ for all α can be written as a sum of the α with nonnegative coefficients. The result now follows from the following lemma, letting the α be the e_i, λ be f, and $\mathfrak{s}$ be L. $\square$

Lemma 3.3. *Let V be a (finite dimensional) $\mathbb{R}$-vector space, $\{e_i\}$ a basis for V^*, and $Q \subset V$ the cone (quadrant) in V defined by $e_i(v) \geq 0$ for all i.*

1. *Let $w \in Q^\circ$ and $L = \mathbb{R}w$. Then the only $f \in V^*$ such that $f|Q \geq 0$ and $f|L = 0$ is $f = 0$.*
2. *Let $W \subset V$ be a linear subspace such that $W \cap Q^\circ = \emptyset$. Then there exists a $f \in V^*$ such that $f|Q \geq 0$, $f|W = 0$, but $f \neq 0$.*

Proof. This is elementary linear algebra. If $W \subset \ker e_i$, then take $f = e_i$. Otherwise, up to a linear transformation on V preserving Q (given in terms of the e_i by an invertible matrix with nonnegative coefficients), we can suppose that $w = \sum e_i^*$, and that $W = L^\perp$. Then in part 1, $f \in Q^*$ and $f|L$ is (a nonnegative multiple of) the sum of the coefficients of f in terms of the basis e_i, and this is positive unless $f = 0$. In part 2, let $f = \sum e_i$ and then $f|W = 0$ by construction. $\square$

Corollary 3.4. *All minimal epimorphic T-normalized subgroups of SL_n are solvable of unipotent dimension $n-1$ and total dimension n.*

It is clear that no such subgroup can have lower dimension, but not obvious without the above classification that all minimal ones have precisely this dimension.

4. THE TITS BOUNDARY AND THE SIZE OF $\mathcal{C}(\Psi)$

4.1. **Minimal** U_Ψ**.** We canonically embed both $\mathcal{G}(\Psi)$ and $\mathcal{C}(\Psi)$ in P^{n-2} via the Tits boundary at infinity of $T < SL_n$ (i.e., the apartment corresponding to $\mathfrak{t}$ in the Tits building of SL_n). First, we embed in $\mathcal{B}' = S^{n-2}$, the space of directions "at infinity" in the root space $\hat{\mathfrak{t}} = \Phi(T,G)\otimes\mathbb{R}$, which is dual to $\mathfrak{t}_\mathbb{R}$ via the standard pairing of $\alpha \in \Phi(T,G)$ with $H_\alpha \in \mathfrak{t}$. Thus $\mathcal{B}' = \hat{\mathfrak{t}}/\mathbb{R}^+ = \mathfrak{t}_\mathbb{R}/\mathbb{R}^+$. Denote either projection modulo $\mathbb{R}^+$ by π.

In particular, following the notation of [FH], we consider the functionals $L_i : \mathfrak{t}_\mathbb{R} \to \mathbb{R}$ which pick out the ith diagonal entry in $\mathfrak{h}$. The $\pm L_i$ form the vertices of two intersecting $(n-1)$-simplices centered at the origin in $\hat{\mathfrak{t}}$, and the positive roots (with respect to B) are the pairwise differences $L_i - L_j$ with $i < j$.

Now, the subset $\mathcal{C}(\Psi) \subset \mathfrak{t}_\mathbb{R}$ projects to a union of Weyl chambers "at infinity" in $\mathcal{B}'$, together with their common walls. In fact, $\mathcal{C}(\Psi)$ must be convex since it arose via a linear transformation from a cone (Q in the proof of Proposition 3.2). A choice of one-dimensional torus in Proposition 3.2 corresponds to the choice of one point in $\mathcal{C}(\Psi) \subset \mathcal{B}'$.

To embed $\mathcal{G}(\Psi)$ in $\mathcal{B}'$, proceed as follows. If ij is an edge with $i < j$, then map it to the edge (wall) connecting $\pi(L_i)$ with $\pi(-L_j)$ in $\mathcal{B}'$. By the distribution of the L_i in $\hat{\mathfrak{t}}$, this realizes $\mathcal{G}(\Psi) \subset \mathcal{B}'$ in such a way that

1. For each i, exactly one of $\pi(\pm L_i)$ is in the image, depending on the parity of the path from i to n in the tree.

2. $\pi(-L_n)$ is in the image, as opposed to $\pi(+L_n)$.
3. No edges cross, and the image is (topologically) a tree homeomorphic to $\mathcal{G}(\Psi)$.

Proposition 4.1. *In $\mathcal{B}'$, $\mathcal{C}(\Psi)$ is the intersection of the (open) hemispheres centered at the midpoints of the edges of $\mathcal{G}(\Psi)$.*

Proof. The midpoints of the edges are precisely the images in $\mathcal{B}'$ of the $\alpha \in \Psi$. The half-space $\{t \in \mathfrak{t} \,|\, \alpha(t) > 0\}$ is bounded by the hyperplane $\alpha = 0$ and contains the image of α under the duality of $\mathfrak{t}_{\mathbb{R}}$ and $\hat{\mathfrak{t}}$. Under projection to $\mathcal{B}'$, these half-spaces become hemispheres, and by construction $\mathcal{C}(\Psi)$ is their intersection. $\square$

Now, observe that a torus $\mathfrak{s}$ intersects $\mathcal{C}(\Psi)$ iff it intersects $-\mathcal{C}(\Psi)$. $\mathcal{G}(\Psi)$ could be embedded equally well via $\pi(-L_i)$ and $\pi(L_j)$ instead of $\pi(L_i)$ and $\pi(-L_j)$, generating the antipodal image. Thus the picture in $\mathcal{B}' = S^{n-2}$ is actually a two-fold cover of a picture in the *projective* Tits boundary $\mathcal{B} = \mathcal{B}'/\pm = \hat{\mathfrak{t}}/\mathbb{R}^\times = \mathfrak{t}_{\mathbb{R}}/\mathbb{R}^\times$.

We remark that the apartment structure on $\mathcal{B}'$ can be interpreted in terms of parabolic subgroups of SL_n (containing T) reverse-ordered by inclusion. The Weyl chambers correspond to the minimal parabolics, i.e., different Borel subgroups containing T. Under this interpretation, the closure of $\mathcal{C}(\Psi)$ consists of those chambers (and walls) corresponding to the parabolics containing TU_Ψ.

4.2. **Nonminimal U_Ψ.** If TU_Ψ is non-minimal epimorphic (containing T), i.e., $\mathcal{G}(\Psi)$ is not a tree, it is clear that the appropriate definition of $\mathcal{C}(\Psi)$, in order to parametrize subtori S such that SU_Ψ is still epimorphic, is as follows: $\mathcal{C}(\Psi) = \bigcup \mathcal{C}(\Psi')$, the union over all Ψ' of Ψ such that $\mathcal{G}(\Psi')$ is a subtree of $\mathcal{G}(\Psi)$.

The embedding of $\mathcal{G}(\Psi)$ into $\mathcal{B}$ is more complicated, since in general the "parity" of a vertex in the graph $\mathcal{G}(\Psi)$ is not well-defined. However, one can still define an object in $\mathcal{B}'$ based on $\mathcal{G}(\Psi)$ by mapping an edge ij with $i < j$ into the *pair* of walls in $\mathcal{B}'$ which join $\pi(L_i)$ with $\pi(-L_j)$, and $\pi(-L_i)$ with $\pi(L_j)$, respectively. The image of this mapping consists of two intertwined image graphs which are antipodes of each other, and which do not intersect except possibly on some vertices. Because of the "parity" issue, the two image graphs in $\mathcal{B}'$ are not necessarily homeomorphic to $\mathcal{G}(\Psi)$. However, the image in the projection $\mathcal{B}' \to \mathcal{B}$ *is* homeomorphic to $\mathcal{G}(\Psi)$, i.e. is a geometric realization. The remaining claims in the following Theorem are now trivial.

Theorem 4.2. *The graph $\mathcal{G}(\Psi)$ of the unipotent part of a T-normalized epimorphic solvable subgroup $H = SU_\Psi$ has a realization in the projective Tits boundary $\mathcal{B}$ of T. The possible subtori S are exactly those which nontrivially intersect $\mathcal{C}(\Psi)$ at infinity. The images of both $\mathcal{G}(\Psi)$ and $\mathcal{C}(\Psi)$ in $\mathcal{B}$ are the unions of the corresponding images for all T-normalized epimorphic solvable subgroups of G which lie in H.*

Note that Proposition 4.1 no longer directly applies when $\mathcal{G}(\Psi)$ is not a tree; we must do the union-over-subtrees construction defined above. However, this construction can be done in either lift of $\mathcal{G}(\Psi)$ to $\mathcal{B}'$ and hence $\mathcal{C}(\Psi)$ is given by the purely geometric data of $\mathcal{G}(\Psi) \subset \mathcal{B}$.

This completes the proof of (a slightly more general form of) Theorem 1.1, and thus of all the results stated in Section 1. Henceforth, we consider $\mathcal{G}(\Psi)$ and $\mathcal{C}(\Psi)$ as realized (i.e. embedded) in $\mathcal{B}$, unless we specify that we mean the *abstract* graph $\mathcal{G}(\Psi)$, which is as in Figure 1.

4.3. **Example: SL_3 and SL_4.** Figure 2 displays $\pm\mathcal{G}(\Psi)$ and $\pm\mathcal{C}(\Psi)$ embedded in $\mathcal{B}'$ for the unique tree type for $n = 3$, numbered $1 — 3 — 2$. In this special case, $\mathcal{G}(\Psi) = \mathcal{C}(\Psi)$, and is indicated by thick lines in the Figure. Weyl chambers are drawn by straight lines and the numbers $\pm k$ indicate the images of $L_{\pm k}$ in $\mathcal{B}'$. The roots α_{12} and α_{13} determining $\mathcal{C}(\Psi)$ are also shown.

Figure 3 displays $\mathcal{G}(\Psi)$ and $\mathcal{C}(\Psi)$ embedded in $\mathcal{B}'$ for the two types of tree corresponding to minimal epimorphic subgroups of SL_4, numbered as in Figure 1. We are looking at a hemisphere in $\mathcal{B}'$ centered at the maximal root α_{14} corresponding to the standard choice of the upper-triangular Borel subgroup B. The thin lines divide this hemisphere into (straightened) Weyl chambers, and those marked with a $\star$ form $\mathcal{C}(\Psi)$. The thick lines indicate (a lift of) $\mathcal{G}(\Psi)$. The numbers are as in Figure 2, but individual roots are omitted.

Weyl conjugation of course permutes the vertex numbers (equivalently translates the diagrams in $\mathcal{B}$), and the "shape" of $\mathcal{C}(\Psi)$ depends only on the tree isomorphism type of $\mathcal{G}(\Psi)$.

Now consider the nonminimal epimorphic solvable subgroup obtained by taking the union of the two trees in Figure 3, i.e., where the abstract graph $\mathcal{G}(\Psi)$ has 4 joined to 1, 2, and 3, and also 2 joined to 3. Figure 4 shows one of the two lifts into $\mathcal{B}'$ of $\mathcal{G}(\Psi)$ and $\mathcal{C}(\Psi)$. We remark that while the abstract graph $\mathcal{G}(\Psi)$ and its realization in $\mathcal{B}$ have a ring formed by vertices 2, 3, and 4, the lifts of $\mathcal{G}(\Psi)$ in $\mathcal{B}'$ do not.

4.4. **Size of $\mathcal{C}(\Psi)$.** By the size of $\mathcal{C}(\Psi)$ we mean the fraction of "random" one-dimensional tori S which, for a given Ψ, will generate an epimorphic subgroup SU_Ψ. This is equal to the proportion of $\mathcal{B}$ covered by $\mathcal{C}(\Psi)$. We see from Figure 3 that this depends on the tree type of $\mathcal{G}(\Psi)$ (graph type if Ψ is not minimal). Loosely speaking, this proportion is determined by the "compactness" of $\mathcal{G}(\Psi)$ in $\mathcal{B}$.

Corollary 4.3. *$\mathcal{C}(\Psi)$ is the largest (among minimal epimorphic subgroups) when $\mathcal{G}(\Psi)$ is the hub-and-spokes tree and the smallest when $\mathcal{G}(\Psi)$ is the straight-line tree.*

Proof. We investigate how much of $\mathcal{B}'$ is thrown away by each edge in Proposition 4.1. Number the edges of $\mathcal{G}(\Psi)$ in some order $e_1, \dots, e_{n-1}$, such that all the subgraphs $e_1, \dots, e_i$ are connected. Let $\mathcal{C}_i$ be the intersection of the hemispheres determined by $e_1, \dots, e_i$, so that $\mathcal{C}(\Psi) = \mathcal{C}_{n-1}$. Clearly, the reduction in size from $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ is minimal exactly when the midpoint of e_{i+1} is the closest possible to the midpoints of $e_1, \dots, e_i$, which happens precisely when all the e_i share a common vertex, i.e., the hub-and-spokes tree.

The reduction in size is maximal when each e_i shares a vertex with only one of the previous e_i's, which is the straight-line tree.

□

FIGURE 2. Tits boundary diagram of the minimal epimorphic T-normalized subgroup of SL_3

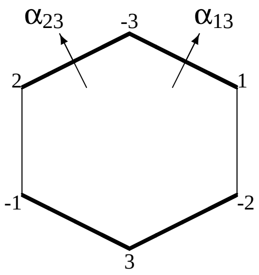


FIGURE 3. Tits boundary diagrams for minimal epimorphic T-normalized subgroups of SL_4

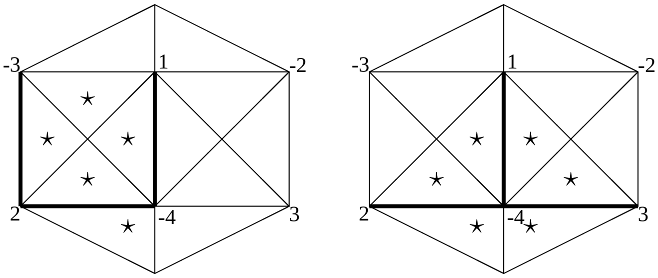
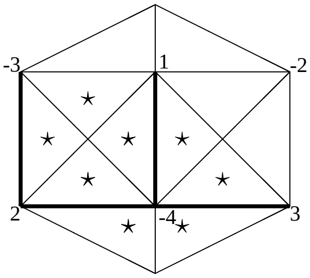


FIGURE 4. Tits boundary diagram of a nonminimal epimorphic T-normalized subgroup of SL_4



The size of $\mathcal{C}(\Psi)$ is of course larger when Ψ is not minimal. When U_Ψ is the full upper-triangular group, S may be "almost any" torus. The only exception occurs when S is in the kernel of the longest root. In this case, we obtain the Borel subgroup of the quasi-parabolic subgroup $Q_{1n} = \langle \ker_T \alpha_{1n}, U_\lambda \,|\, (\lambda, \alpha) \geq 0 \rangle$, and Q_{1n} is the largest subgroup in which this Borel subgroup is epimorphic. In fact, such quasi-parabolic subgroups are precisely the maximal nonreductive observable[3] subgroups of SL_n (see [Suk] or [Gro, Sections 3 and 7]).

We see from Figures 2 and 3 that the size of $\mathcal{C}(\Psi)$ is $2/3$ for the unique tree type in SL_3, and $5/12$ and $1/2$ for the straight-line and hub-and-spokes trees respectively in SL_4.

5. EXTENSIONS

5.1. **Other semisimple groups.** The principles behind the constructions in this paper apply equally well to other semisimple groups G than SL_n. The difference is that more complicated Dynkin diagrams translate into different edge- and vertex-labellings, as well as connectedness conditions, on the (abstract) graph $\mathcal{G}(U)$, now best considered as some sort of more complicated combinatorial object. Thus the graph-theoretic side of Theorem 1.1 is more complicated, and less useful, to articulate. We can still, however draw various qualitative conclusions:

1. As long as some simple factor of G has rank ≥ 3, there are numerous conjugacy types of unipotent radicals of minimal solvable $H <_e G$, and the number of types increases with rank.
2. The freedom in choosing subtori S, such that SU_Ψ is epimorphic, varies depending on the "compactness" of a graph-theoretic representation of Ψ. There are specific "most-compact" and "least-compact" minimal types (corresponding to the hub-and-spokes and straight-line trees for SL_n).

If G is a product of simple factors, the graph $\mathcal{G}(\Psi)$ will have several connected components, but the torus structure of T-normalized (rather than T-containing) subgroups is complicated by the nonsimplicity. This is shown by the example [Wei], $H = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & a^{-1} \end{pmatrix} < SL_2 \times SL_2$.

5.2. **Normalized-generated epimorphic subgroups.** Already in SL_3, there is an (essentially unique) non-T-normalized minimal epimorphic solvable subgroup ([Moz]), whose Lie algebra is generated by

$$D = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \; X = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

Here, D and X generate the (epimorphic) Borel subalgebra $\mathfrak{b}$ in a copy $\mathfrak{j}$ of $\mathfrak{sl}_2$ such that $\mathfrak{j}$ and Z generate $\mathfrak{sl}_3$.

Larger non-T-normalized examples can be generated in higher-rank semisimple G by replacing J by higher-rank semisimple subgroups G' and $\langle \exp sD, \exp tX \rangle$

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[3] $H < G$ is observable if it is the stabilizer of a vector in some linear representation of G, equivalently if the largest $H < L < G$ such that $H <_e L$ is $L = H$ itself.

by an epimorphic subgroup of G' normalized by a maximal torus of G' (but not necessarily a parabolic). This has prompted us (see [Per, Sections 3.2 and 3.4]) to make the following

Definition 5.1. Suppose G is a reductive Lie group and $H <_e G$. Call H *ng-epimorphic* (for *normalized generated*) if there exist subgroups G_i, $i = 0 \ldots k$, such that $\langle G_i \rangle_i = G$, $G_0 < H$, and for $i \geq 1$ each G_i is semisimple and $G_i \cap H$ contains an epimorphic subgroup of G_i normalized by a maximal torus of G_i.

The results in this paper reduce the classification of *ng*-epimorphic subgroups to that of generation of semisimple groups by semisimple subgroups, which depends only on G. We have conjectured that all epimorphic subgroups of semisimple groups are *ng*-epimorphic and proved it for low rank ([Per, Conjecture 3.12, Section 4]). All higher-rank examples known to the author from the literature are also *ng*-epimorphic.

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