Point stabilizers of connection preserving actions

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Abstract

Suppose G is a real algebraic group. We investigate which algebraic subgroups can arise as point stabilizers in affine connection preserving actions on manifolds, and which subgroups have the property that their fixed points in such actions are necessarily fixed by all of G.

Subgroups with analogous properties concerning invariant vectors of (finite-dimensional) linear representations are called observable and epimorphic, and are of interest in representation theory. We prove that under certain hypotheses (conjecturally always) the classes of subgroups with the respective properties coincide.

Techniques involve contrasting local dynamics and linearization of the stabilizer representation of a connection-preserving action, and the structure theory of epimorphic subgroups.

Contents

0.1 Background

Suppose G acts on a manifold M, preserving some geometric structure ω. The Zimmer program [Zim2] aims to relate (1) the group structure of G, (2) the topology of M, (3) the nature of the structure ω, and (4) the dynamics of the action. Typically, G is a semisimple Lie group or a discrete subgroup of such. Examples of geometric structures include Riemannian or pseudo-Riemannian metrics, invariant measures, connections, and generalizations. Considerable progress has been made by many researchers under various hypotheses. The study of certain isotropy subgroups has proven crucial, but isotropy remains poorly understood in general. This paper provides some results on isotropy subgroups when the group action preserves an affine connection.

Suppose $H < G$, where G is an algebraic group (for us, over $\mathbb{R}$ or $\mathbb{C}$). H is called *observable* if it is the stabilizer of a vector in some finite-dimensional linear representation of G. H is called *epimorphic* if any vector in a finite-dimensional linear representation of G which is fixed by H is necessarily fixed by all of G. These are "opposite" properties. Observable and epimorphic subgroups have been extensively studied; in particular, observable subgroups have (essentially) been classified.

By analogy with the above, suppose G is a (real) Lie group. Let us call $H < G$ *c-observable* (for *connection*-observable), if H is the stabilizer of a point in a G-action on some manifold M preserving an affine connection, and call $H < G$ *c-epimorphic* if any point fixed by H in such a G-action is necessarily fixed by all of G.

We conjecture the following

Conjecture 0.1 (Desired result). *Suppose G is any real algebraic group, and H an algebraic subgroup. Then $H < G$ is c-observable (resp. c-epimorphic) iff it is observable (resp. epimorphic).*

In his 1986 I.C.M. address [Zim1], Zimmer describes the program stated in the first paragraph of this paper from a slightly different point of view. One generally studies a group G by examining its representations (or realizations) in some natural class $\mathcal{C}$ of groups. For instance, if $\mathcal{C}$ consists of endomorphism groups $GL(V)$ of finite-dimensional vector spaces, one obtains finite-dimensional linear representation theory. Infinite-dimensional unitary representation theory is another example. In the Zimmer program, one considers a form of "geometric representation theory", where $\mathcal{C}$ consists of automorphism groups $\mathrm{Aut}(M,\omega)$ of various geometric structures on manifolds. The aim is, of course, to discover interesting new information on G, and to uncover parallels between the various representation theories. The above Conjecture lies naturally in this context.

0.2 Statement of results

The results in this paper include the following cases of the Conjecture.

Theorem 0.2. *Suppose G is reductive. Then $H < G$ is epimorphic iff it is c-epimorphic, under either of the following conditions:*

1. *(H is big) H is normalized by a maximal split torus of G.*
2. *(G is small) $G/\mathcal{R}G$ is a product of compact, $\mathbb{R}$-rank 1, and the following $\mathbb{R}$-rank 2 factors: $SL_3(\mathbb{R}), Sp(4,\mathbb{R}), SO(p,2), SU(p,2), Sp(p,2)$, or G_2 ($p \geq 2$).*

Theorem 0.3. *Suppose G is reductive. Then $H < G$ is observable iff it is c-observable, under either of the following conditions:*

1. *(H is big) H is of full rank in G (contains a maximal split torus of G).*
2. *(G is small) $G/\mathcal{R}G = G/Z(G)$ is a product of $\mathbb{R}$-rank 1 or $SL_3(\mathbb{R})$ factors.*

The arguments actually apply under more general hypotheses which in particular show that Conjecture 0.1 is true for all known examples of epimorphic subgroups of semisimple G, and that two other conjectures on the structure of epimorphic groups would prove Conjecture 0.1 in full generality.

0.3 Outline of paper and methodology

We start (Section 1) with basic properties of epimorphic and observable subgroups. Then we study the local dynamics of SL_2 actions on manifolds in Section 2, with a view to consequences for higher-rank semisimple groups. The principal results of this section are Proposition 2.5, which shows that the Borel subgroup of SL_2 is c-epimorphic by playing off local dynamics versus linearity; and Theorem 2.9, which shows that even nonlinear actions of SL_2 behave like linear representations in a certain way, if merely their restriction to the upper unipotent subgroup is linear.

In Section 3.2 we combine these results with the structure theory of epimorphic subgroups of semisimple groups to prove the "big H" case of Theorem 0.2 with the extra hypothesis of semisimplicity (with no compact factors) on G. In the rest of Section 3, we perform a series of Lie algebraic reductions to remove the semisimplicity hypothesis on G, relax the "big H" hypothesis to a type of epimorphic subgroup we call *ng*-epimorphic, and deduce Theorem 0.3 from Theorem 0.2. We conjecture that all epimorphic subgroups are *ng*-epimorphic (which would imply Conjecture 0.1 in full generality), and prove that this is so under the "small G" hypotheses in Section 4 and Appendix A.

Throughout, the only relevant feature of affine connection preserving actions is that the action of an isotropy subgroup G_m linearizes in some neighbourhood of the fixed point m. The difference from the linear-representation picture is that the linearization (i) is not global in TM_m, and (ii) does not extend to all of G. Since the question is local, there is no need for the ergodic techniques typical of parts of the Zimmer program, and thus hypotheses on the existence of an invariant measure or on G having no compact factors are not necessary.

Somewhat analogous results, dealing with actions of epimorphic subgroups of G on homogeneous spaces L/Γ, where $L > G$ and $\Gamma < L$ is a lattice, and with an invariant measure rather than a connection, have recently been obtained by S. Mozes, N. Shah, and Barak Weiss [Moz, SW, Wei], using largely different techniques.

0.4 Examples

As orientation for the reader, we indicate a few (nonexhaustive) examples to which our results apply in the case $G = SL_3(\mathbb{R})$.

Example 0.4. *The following are epimorphic and c-epimorphic in $SL_3(\mathbb{R})$:*

1. *H any parabolic subgroup.*

2. $H = \left\{ \begin{pmatrix} a & 0 & b \\ & a & c \\ & & 1/a^2 \end{pmatrix} \right\}$

3. $H = \left\{ \begin{pmatrix} a & b & c \\ & 1 & b \\ & & 1/a \end{pmatrix} \right\}$

Example 0.5. *The following are observable and c-observable in $SL_3(\mathbb{R})$:*

1. *H any nilpotent, diagonalizable, or reductive subgroup.*

2. $H = \left\{ \begin{pmatrix} 1 & b & c \\ & a & d \\ & e & 1/a \end{pmatrix} \right\}$

3. $H = \left\{ \begin{pmatrix} a & b & c \\ & 1/a^2 & d \\ & & a \end{pmatrix} \right\}$

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1 Preliminaries

1.1 Epimorphic and observable subgroups over $\mathbb{R}$ or $\mathbb{C}$

The notions of observable and epimorphic subgroups were introduced by Bialynicki-Birula, Hochschild, Mostow, and Bergman in the 1960s and 1970s. More recent work has been done by Bien and Borel (e.g., [BB1, BB3]) and Grosshans [Gro], which are also comprehensive references.

We denote H observable (resp. epimorphic) in G by $H <_o G$ (resp. $H <_e G$). The following properties are well-known and will be used freely in the sequel. We concentrate on epimorphic subgroups, since they will drive the argument.

1. If $H < H' < G$ and $H <_e G$ then $H' <_e G$. If $H <_e H' <_e G$ then $H <_e G$
2. $H <_e G$ iff the Zariski closure of H is epimorphic in G.
3. Parabolic subgroups are epimorphic. Solvable groups have no nontrivial epimorphic subgroups.
4. If $H <_e G$ and $G' < G$, then $H \cap G'$ may not be epimorphic in G'. In particular, counterexamples exist when $G' = \mathcal{R}_u G$ or $\mathcal{R} G$ (even when G is reductive), or when G' is a simple factor of a semisimple G.
5. However, if $G = \langle G_i \rangle_i$ and $H \cap G_i <_e G_i$ for each i, then $H <_e G$. ($\langle \cdot \rangle$ denotes the group generated by the indicated subgroups or subsets.)
6. Epimorphicity is preserved forwards but not backwards under group morphims, e.g., projections.
7. If $H <_o H' <_o G$ then $H <_o G$. $H <_o G$ iff $\mathcal{R} H <_o G$.
8. $H <_o G$ iff G/H is quasi-affine, i.e., is an open subset of an affine variety.

9. $H <_o G$ iff any finite-dimensional rational representation of H can be extended to a finite-dimensional representation of G, i.e., every finite-dimensional rational H-module is an H-submodule of a finite-dimensional rational G-module.

1.2 Epimorphic and observable subgroups over $\mathbb{R}$

Suppose G is a real algebraic group and H an algebraic subgroup. It is shown in [BBHM, Theorems 5 and 6] that $H <_o G$ iff $H \otimes \mathbb{C} <_o G \otimes \mathbb{C}$ iff $H^0 <_o G^0$. By [BB3, Prop. 3.2 and 1.6], the same is true replacing "observable" with "epimorphic".

Furthermore, to test for $H <_o G$ or $H <_e G$, where H and G are defined over any extension of $\mathbb{Q}$, it suffices to consider $\mathbb{Q}$-representations of $G_{\mathbb{Q}}$, and hence the notation $<_o$ and $<_e$, without specifying a base field, is unambiguous.

Over $\mathbb{C}$, any $H <_e G$ trivially contains a solvable $H' <_e G$, namely the Borel subgroup of H. Thus the study of epimorphic subgroups over $\mathbb{C}$ reduces to the study of solvable ones. Over $\mathbb{R}$, the situation is more complicated, but if G is semisimple with no compact factors, we may restrict to studying $\mathbb{R}$-split solvable subgroups by the following:

Proposition 1.1. *(Bien-Borel [BB1, BB3]) If $H <_e G$ are complex algebraic groups, k is a subfield of $\mathbb{C}$, and G is generated by its unipotent k-subgroups, then there exists an k-split solvable $H' < H$ such that $H' <_e G$.*

1.3 Observable hull

The following demonstrates how observability and epimorphicity are opposite and complementary properties. A proof can be found in any of the general references mentioned above.

Proposition 1.2 (Observable hull). *Suppose $H < G$, over $\mathbb{C}$ or $\mathbb{R}$. There is a unique Zariski-closed subgroup L, called the observable hull of H, such that $H <_e L <_o G$. L is minimal such that $H < L <_o G$ and maximal such that $H <_e L < G$. L is defined over the same base field as H and G.*

1.4 T-normalized epimorphic subgroups of semisimple Lie groups

The structure of T-normalized epimorphic subgroups of semisimple Lie groups over $\mathbb{C}$ has been studied by Pommerening, and Bien and Borel; the T-normalization hypothesis allows productive use of the language of roots and weights.

Proposition 1.3 ([Pom], also [BB1, BB3] and [Gro, Lemma 3.11]).
Suppose $H <_e G$, where G is a semisimple Lie group over $\mathbb{C}$. Suppose H is normalized by T, and assume without loss of generality that H is solvable. Then $H = SU$ where S is a subtorus of T and $U = U_\Psi$ is a unipotent group whose Lie algebra is a direct sum of root spaces of T, namely $\mathfrak{u} = \bigoplus_{\alpha\in\Psi} \mathfrak{n}_\alpha$, such that

1. *$\langle T, U, U^- \rangle = G$; equivalently any root of G with respect to T is an integral linear combination of the $\alpha \in \Psi$; and*
2. *the only character λ of T, such that $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Psi$ and such that $\lambda|S = 0$, is $\lambda = 0$.*

Conditions 1 and 2 also imply epimorphicity of H, the principal ingredient in the proof that condition 2 on S is sufficient being Corollary 2.8 in this paper.

Our companion paper [Per] deals with Proposition 1.3 in more detail, in particular sharpening it to obtain a classification of all T-normalized epimorphic subgroups of SL_n. (There are many nonconjugate classes, along the lines of Example 0.4 (1) and (2), for $n > 3$.) For our present purposes, however, it suffices to observe that in light of Proposition 1.1, the same conclusions apply in the case where G is a semisimple Lie group over $\mathbb{R}$ with no compact factors.

Corollary 1.4. *Assume the set-up of Proposition 1.3, and let T' be the subtorus of T with Lie algebra $\mathfrak{t}' = \langle [\mathfrak{n}_\alpha, \mathfrak{n}_{-\alpha}] \rangle_{\alpha\in\Psi}$. Then $\mathfrak{t} = \langle \mathfrak{t}', \mathfrak{s} \rangle$ and $\langle S, U, U^- \rangle = G$.*

Proof. It suffices to prove the first conclusion. Suppose D belongs to the orthogonal complement of $\langle \mathfrak{t}', \mathfrak{s} \rangle = \mathfrak{t}' + \mathfrak{s}$ in $\mathfrak{t}$. Let λ be the character of T dual under $(\cdot,\cdot)$ to D. By the choice of D, we have $\lambda|S = 0$ and $(\lambda, \alpha) = 0$ for all $\alpha \in \Psi$. But then λ and hence D are 0, and so the conclusion is true. □

1.5 Classification of observable subgroups over $\mathbb{C}$

Let Q be stabilizer of a highest-weight vector in some irreducible representation of G; such Q are called quasi-parabolic (taking the trivial representation allows $Q = G$). It is clear that $Q <_o G$ and furthermore any group $L < Q$ such that $\mathcal{R}L < \mathcal{R}Q$ is observable in G. Such L are called subparabolic.

Proposition 1.5 ([Suk],[Gro, Sections 3 and 7]). *All observable subgroups of a reductive group are subparabolic.*

Quasi-parabolics are in one-to-one correspondence with characters $\chi \in X(T)$ of a maximal torus. The quasiparabolic corresponding to χ is

$$Q_\chi = \langle T_\chi, U_\alpha \,|\, (\chi, \alpha) \geq 0 \rangle, \text{ where } T_\chi = \ker \chi.$$

We only make incidental use of this classification, in Section 4.4.

1.6 Affine connections and actions

We will assume all group actions to be C^2, i.e., their infinitesimal generators are C^1 in local coordinates. All connections in this paper are affine, even if the word "affine" is omitted.

Proposition 1.6. *Suppose any group G acts on a manifold M preserving an affine connection. If $m \in M$ is fixed by G, the action locally linearizes, i.e., there is a neighbourhood Ω of m and a local coordinate chart $\phi : \Omega \to TM|_m$ under which the G-action is the same as the induced stabilizer representation π on $TM|_m$. This means that the diagram*

$$\begin{array}{ccc} \Omega & \xrightarrow{g\cdot} & \Omega \\ \phi\downarrow & & \downarrow\phi \\ TM|_m & \xrightarrow[\pi(g)]{} & TM|_m \end{array}$$

commutes whenever it makes sense.

The local coordinates in the Proposition are given by the exponential map defined by the connection. For details, see [Sza, Section 3.1].

1.7 Immediate consequences

Two of the four directions in Conjection 0.1 are immediate:

Proposition 1.7. *Suppose G is any $\mathbb{R}$-algebraic group.*

1. *If $H <_o G$ then H is c-observable.*
2. *If $H < G$ is c-epimorphic, then $H <_e G$.*

Proof. If H is observable, it is the stabilizer of a vector v in some linear representation of G on $M = \mathbb{R}^n$ for some n. Since the G action is linear, it preserves the flat connection on M.

For the second claim, since linear actions on $\mathbb{R}^n$ preserve the canonical connection, c-epimorphicity is clearly at least as stringent a condition as epimorphicity. □

We remark that if $H <_o G$, we can actually construct a compact manifold on which G acts preserving a connection, such that H is a point stabilizer. Choose $M = \mathbb{R}^n$ as in the above proof and denote by μ_x scalar multiplication by a positive real number x on M. Let $M_1 = (M - \{0\})/\mu_2$ and $M_2 = (M - \{0\})/\mu_\pi$. These are compact manifolds on which G acts preserving a connection, with points $m_i \in M_i$ with stabilizer $H_1 = \langle H, \mu_2 \rangle$ and $H_2 = \langle H, \mu_\pi \rangle$. Now let G act on $M_1 \times M_2$; the stabilizer of (m_1, m_2) is H as desired.

The question of when H arises as the point stabilizer in a connection-preserving action on a *transitive* $M (= G/H)$ was considered by several authors in the 1950s (see [KN, vol II, p. 190-200]). It was noted that a sufficient condition is for H to be *reductive in* G, i.e. for $\mathfrak{h}$ to have an $\mathrm{Ad}(H)$-invariant complement $\mathfrak{m}$ in $\mathfrak{g}$. In this case, the invariant affine connections on G/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant bilinear forms $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. This sufficient condition includes in particular the case when H is a reductive group (the simplest case in the Sukhanov classification) but does not include the case of H being a quasiparabolic group, for instance the standard example of the stabilizer of $(0, \ldots, 0, 1)$ in the standard transitive action of $SL_n(\mathbb{R})$ on $\mathbb{R}^n - \{0\}$.

Finally, we also note that the observable hull lets us concentrate on Theorem 0.2 and prove Theorem 0.3 as a consequence: Suppose H is c-observable (i.e., is a point stabilizer in an appropriate action) and L its observable hull. Then $H <_e L$ and thus, provided the relevant hypotheses apply to L, c-epimorphic in L. But since H is the actual point-stabilizer, $H = L$ and so $H <_o G$.

2 SL_2 actions

In this section, we obtain restrictions on $SL_2(\mathbb{R})$ actions with locally linearizable stabilizers by playing off the linearization versus the local dynamics of the action. We will later use these results to draw conclusions about the local dynamics of higher-rank actions from examining embedded copies of SL_2.

Throughout, let J be SL_2, and KAN be the usual Iwasawa decomposition of J with $B = AN$ the upper triangular Borel subgroup. N^- is the lower triangular unipotent subgroup, opposite to N. When necessary, we label the individual elements of the standard one-parameter subgroups as follows:

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a_s = \begin{pmatrix} e^s & \\ & e^{-s} \end{pmatrix}, \quad \text{and} \quad u_t = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$

2.1 Model actions

We will consider J-actions modelled on the following two standard examples.

Example 2.1 (Orbit is J/N). *Consider the standard SL_2-action on $\mathbb{R}^2 - \{0\}$, and let D, X, and Y be the standard generators of $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{n}^-$ corresponding to a_s, u_t,* *and the "lower-triangular opposite" of u_t respectively. The infinitesimal generators for this action have the form*

$$D^{\#} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad X^{\#} = y\frac{\partial}{\partial x}, \quad \text{and} \quad Y^{\#} = x\frac{\partial}{\partial y}.$$

The action is transitive and the stabilizer of a point $m = (x_0, 0)$ is N. The orbit Am is the x-axis and is fixed pointwise by N.

Suppose we now transform coordinates in $\mathbb{R}^2$ by a diagonal matrix Q. In the new coordinates, the action of the matrices D, X, Y is by their respective conjugates by Q. This amounts to scaling $X^{\#}$ by a some constant C, scaling $Y^{\#}$ by $1/C$, and leaving $A^{\#}$ unchanged. Conversely, if $D^{\#}, X^{\#}, Y^{\#}$ are of this form for some $C \neq 0$, via a suitable choice of Q we can transform to the case $C = 1$.

Example 2.2 (Orbit is J/B). *Consider the fractional linear action of SL_2 on the circle $S^1 = \mathbb{R} \cup \{\infty\}$, given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}.$$

This action is again transitive, and the stabilizer of the point $m = \infty$ is B. The N-action on the rest of orbit is by translation on $\mathbb{R}$. In particular, if $x \neq m$, then as $t \longrightarrow \pm\infty$, $u_t x \longrightarrow m$.

2.2 B-fixed points are J-fixed points

Lemma 2.3. *Suppose $\pi : B \longrightarrow GL(V)$ is any continuous, finite dimensional representation. Then each $\pi(u_t)$ is unipotent.*

Proof. If π is assumed to be algebraic (rational), this follows from the preservation of Jordan decomposition under morphisms of affine groups [Bor, I.4.4(4)]. However, the hypothesis of rationality is not necessary, since $a_{-s}u_t a_s = u_{e^{-2s}t} \longrightarrow \mathrm{Id}$ as $s \longrightarrow \infty$, a_s contracts u_t. Thus $\pi(a_{-s})\pi(u_t)\pi(a_s) \longrightarrow \mathrm{Id}$ and hence $\pi(u_t)$ is contracted as well. So all the $\pi(u_t)$ have all eigenvalues 1 and are unipotent. □

Lemma 2.4. *Let $\pi(u_t)$ be a one-parameter unipotent representation on a finite dimensional vector space V. If $\pi(u_t)v \longrightarrow 0$ as $t \longrightarrow \infty$, then $v = 0$.*

Proof. This is elementary linear algebra. Choose a basis for V such that $\pi(u_t)$ is upper triangular. Let $v = (v_1, \ldots, v_n)$ and $\pi(u_t)v = (w_1(t), \ldots, w_n(t))$. Then

$$w_k(t) = v_k + f_k(v_{k+1}, \ldots, v_n; t),$$

where for each t, $f_k(\cdot;t)$ is a linear function on V, and so $f(0,\ldots,0;t)=0$.

Now $w_n(t)=v_n$, so since $w_n(t)\to 0$, we must have $v_n=0$. Thus $w_{n-1}(t)=v_{n-1}+f_{n-1}(0;t)=v_{n-1}$, and since $w_{n-1}(t)\to 0$, v_{n-1} must be 0. Proceeding in this fashion, we see that each $w_k(t)=v_k$ independent of t and hence v_k must equal 0. Thus $v=0$. □

Proposition 2.5. *Suppose J acts on M, $m\in M$ is fixed by B, and the B-action locally linearizes near m. Then m is a J-fixed point.*

Proof. Linearize the B action near m i.e. choose coordinates in some neighbourhood Ω of m such that the B action is given by the $\pi(B)$ action on TM_m, where π is the stabilizer representation.

Suppose $g\in J$ does not fix m and let $x=gm$ with $g\in K$. The orbit $Jm=Jx$ is as in Example 2.2, and so as $t\to\pm\infty$, $u_tx\to m$. We shall suppose that for $t>0$, u_tx remains in Ω. Up to replacing u_t by u_{-t}, or x by $u_{t'}x$ for a t' sufficiently large, this is always the case.

Finally, let v be the vector in TM_m identified with x in M by the linearization. The u_t action for $t>0$ on x is the $\pi(u_t)$ action on v. By Lemma 2.3, $\pi(u_t)$ is a unipotent representation and so by Lemma 2.4, $v=0$ and thus $x=m$. □

2.3 Linear representations of J with N-fixed points

Above, we showed that (local) linearity of the N-action near a point m with stabilizer B was incompatible with the dynamics of the N-action on J/B. Now suppose m has stabilizer N. The standard N-action on J/N described in Example 2.1 *is* linear. We need to examine the structure of such actions in more detail.

The utility of this theory will be in analyzing the diagonal part of epimorphic (and c-epimorphic) subgroups of higher-rank semisimple groups and algebras. To facilitate this, we consider actions of the Lie algebra $\mathfrak{j}_\alpha$, determined by generators D,X,Y and bracket relations

$$[D,X]=\alpha X,\quad [D,Y]=-\alpha Y,\quad [X,Y]=D.$$

Here α is positive constant. Each algebra $\mathfrak{j}_\alpha$ is of course isomorphic to the standard $\mathfrak{sl}_2=\mathfrak{j}_2$. However the parameter α arises naturally in embeddings of SL_2 in higher-rank groups.

We start by recalling the (finite-dimensional) representation theory of $\mathfrak{sl}_2$, without proof, modified to account for the parameter α. We view the results somewhat through the eyes of a differential geometer, in view of generalizing them in the next section to actions which are *a priori* only linear restricted to N.

In this and the next section, we work over the base field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.

Proposition 2.6.

1. *The standard linear representation of $\mathfrak{j}_\alpha$ on $E = \mathbb{F}^2$ is given by the matrices*

$$D = \begin{pmatrix} \alpha/2 & \\ & -\alpha/2 \end{pmatrix}, \quad X = \begin{pmatrix} & 1 \\ & \end{pmatrix}, \quad Y = \begin{pmatrix} & \\ \alpha/2 & \end{pmatrix}.$$

The infinitesimal generators are thus (cf. Example 2.1)

$$D^\# = (\alpha/2)x\frac{\partial}{\partial x} - (\alpha/2)y\frac{\partial}{\partial y}, \quad X^\# = y\frac{\partial}{\partial x}, \quad \text{and} \quad Y^\# = (\alpha/2)x\frac{\partial}{\partial y}.$$

The action is transitive and the kernel of the $\mathfrak{n}$-action (the set of N-fixed vectors) is the x-axis, an A-invariant line.

Scaling the x- and y-coordinates does not change D, but multiplies X and Y by constants which are reciprocals of each other.

2. *There is a unique irreducible representation of $\mathfrak{j}_\alpha$ on $E = \mathbb{F}^{n+1}$, which is the nth symmetric power of the standard representation. Ordering and labelling the basis vectors as $e_0 = x^n$, $e_1 = x^{n-1}y$, $\ldots$, $e_n = y^n$, we obtain that the matrix of D is*

$$D = \mathrm{Diag}(n\alpha/2, (n-1)\alpha/2, \ldots, -n\alpha/2).$$

The matrices of X and Y are upper and lower off-diagonals; the precise entries can again be varied by scaling the basis vectors e_k. The infinitesimal generators can be read off as previously.

The kernel of the $\mathfrak{n}$-action is an A-invariant line W, namely the e_0-axis. D acts on W with eigenvalue $n\alpha/2 > 0$.

We remark that in the higher-dimensional irreducible representations, the actual J_α orbit $J_\alpha W$ is a nonlinear 2-dimensional object whose linear span is all of $\mathbb{F}^{n+1}$. The action on this orbit may appear vastly nonlinear in very natural coordinates. This is shown in the following:

Example 2.7. *Consider the standard irreducible linear representation of $\mathfrak{sl}_2$ on $\mathbb{R}^3$, described by the embedding $\mathfrak{j} = \mathfrak{sl}_2 < \mathfrak{sl}_3$ given by*

$$D = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}, \quad X = \begin{pmatrix} & 1 & \\ & & 2 \\ & & \end{pmatrix}, \quad Y = \begin{pmatrix} & & \\ 2 & & \\ & 1 & \end{pmatrix}$$

We have $\alpha = 2$; the disposition of the 2's in X and Y is chosen for symmetry. The A-invariant line W is the x-axis. The J-orbit JW is the right-angle cone which *contains the positive e_0- and e_2-axes. It is described analytically by the equation $e_1^2 = 2e_0e_2$. Off of the e_0-axis, N-suborbits consist of constant-e_2 slices of the cone; A-suborbits are constant-e_1 slices, and N^--suborbits constant-e_0 slices.*

We of course have canonical coordinates on JW given by identification with J/N. However, we also can take as coordinates on the orbit the pair (e_0, e_2) (with $e_1 = \sqrt{2e_0e_2}$). In these coordinates, we have

$$D^\# = 2e_0\frac{\partial}{\partial e_0} - 2e_2\frac{\partial}{\partial e_2},\ X^\# = \sqrt{2e_0e_2}\frac{\partial}{\partial e_0},\ \text{and } Y^\# = \sqrt{2e_0e_2}\frac{\partial}{\partial e_2}.$$

In these coordinates, $X^\#$ and $Y^\#$ are not linear, but $D^\#$ is.

Corollary 2.8 (Pommerening [Pom]). *Let ρ be a finite-dimensional representation of $\mathfrak{j}_\alpha$, and W a A-invariant line fixed pointwise by N. Let λ be the eigenvalue of the D action on W. Then $\lambda \geq 0$, with $\lambda = 0$ iff W is fixed by all of $\mathfrak{j}_\alpha$.*

Proof. The representation ρ must be a direct sum of irreducible representations, and thus W is the span of the e_0 axis of some symmetric power of the standard representation (or possibly in the span of the e_0 axes of several summands in the case of higher multiplicity). In fact, $\lambda = p\alpha/2$, where p is a positive integer strictly less than the dimension of ρ.

We have $\lambda = 0$ iff W is fixed pointwise by AN, in which case it is fixed pointwise by J_α since the parabolic subgroup AN is epimorphic. ☐

Pommerening actually states the above result using the language of weights of higher rank groups, as follows. Using the notation of Proposition 1.3, suppose λ is a T-weight of a representation of G, and W the associated weight space (a T-invariant line with eigenvalue λ). Let $x \in W$ be fixed by U_α. The conclusion is that $(\lambda, \alpha) \geq 0$. Our formulation is equivalent, since (λ, α) is positive iff $\lambda(D_\alpha)$ is. Hence this is actually a rank-one result. Pommerening's proof is via transitivity of the Weyl group.

2.4 N-linear actions of J with N-fixed points

We now generalize the discussion above to actions of J_α ($\mathfrak{j}_\alpha$) which are not linear representations, where we merely know the N-action is linear. Our goal is the following, which essentially states that the action behaves like a linear representation on the kernel (fixed point set) of the N-action:

Theorem 2.9 ("Representation theory" of $\mathfrak{n}$-linear actions of $\mathfrak{j}_\alpha$).

1. *Suppose $\mathfrak{j}_\alpha$ acts transitively on $E = \mathbb{F}^2$ such that N is the stabilizer of a point $m \in E$. Suppose the $\mathfrak{n}$-action on E is linear. Then, up to a translation and rotation of E, m lies on the x-axis; $X^\# = Cy(\partial/\partial x)$; and $D^\#|_{y=0} = (\alpha/2)x(\partial/\partial x)$. By a further scaling of the axes, we can force $C = 1$.*

2. *Suppose $\mathfrak{j}_\alpha$ acts on $E = \mathbb{F}^{k+1}$ such that the $\mathfrak{n}$-action is linear. Let V be the kernel of the $\mathfrak{n}$-action. Then V is an A-invariant linear subspace (after translation to pass through 0) on which $\mathfrak{a}$ acts linearly. Furthermore, the matrix of $D^\#|V$ diagonalizes and its eigenvalues are nonnegative real numbers, of the form $p\alpha/2$ where $0 \leq p \leq k$ is an integer.*

3. *The conclusions of 1. and 2. remain true for the action of J_α on a manifold M (now over $\mathbb{F} = \mathbb{R}$), in local coordinates ϕ in a neighbourhood of an N-fixed point $m \in M$, such that the N action is linear in terms of ϕ.*

Of course, in any $\mathfrak{j}$ action the orbit of a point with stabilizer N can be identified with J/N (c.f. Example 2.1), and in those coordinates the conclusion of part 1 of the Theorem are trivially true. The point is that they remains true in any externally-imposed coordinates in which the $\mathfrak{n}$-action remains linear. Alternatively, part 1 of the Theorem states that we may not transform coordinates on $\mathbb{F}^2 - \{0\} = J/N$ in any way such that both (i) the x-axis is bent or the action on it stretched nonlinearly; and (ii) the global $\mathfrak{n}$ action remains linear.

We start with the following trivial observation, which implies that in any SL_2 action the kernel of the N-action is A-invariant, since A normalizes N:

Lemma 2.10. *Suppose a group G acts on a set S, and a subgroup $H < G$ fixes an element $s \in S$. Suppose $K < G$ normalizes H. Then the whole orbit Ks is pointwise fixed by H.*

Proof. Let $x = ks$ for some $k \in K$. Since K normalizes H, $Hx = Hks = kHs = \{ks\} = \{x\}$. □

Lemma 2.11. *Suppose J_α acts on $\mathbb{F}^2$ such that N is the stabilizer of a point m and acts linearly. Then (up to a rotation and translation), Am lies on the x axis and $X^\# = Cy\frac{\partial}{\partial x}$, for some nonzero constant C.*

Proof. By the previous Lemma, Am is pointwise fixed by N. Being the kernel of a linear action, it a straight line. Rotate and translate so that it is (possibly an interval on) the x-axis. By linearity $X^\#$ is independent of x.

To complete the proof, it suffices to show that $X^\#$ has no $\frac{\partial}{\partial y}$ component. Suppose the contrary. Up to a further linear transformation preserving the x-axis, we can force $X^\#$ to be parallel to the y-axis. Thus the N-orbits are of the form $(x, \mathbb{R}^+)$,

$(x,\mathbb{R}^-)$, or the single points $(x,0)$, which constitute Am. (If $\mathbb{F}=\mathbb{C}$, the first two coalesce into $(x,\mathbb{C}-\{0\})$).

Now let $\psi:\mathbb{F}^2\to A$ be the projection onto the x-axis in the above coordinates. It has the dynamical interpretation $\psi(m)=\lim_{t\to\infty}(\exp tX)m$. In particular, there must be points in Jm lying in $\psi^{-1}(0)$, i.e. $ka\in KA$ so that $\lim_{t\to\pm\infty}u_t ka\in N$. However,

$$\operatorname{trace}(u_t k_\theta a_s)=(e^a+e^{-a})\cos\theta - te^a\sin\theta \not\to 2=\operatorname{trace}(N)$$

unless $ka=\mathrm{Id}$, a contradiction.

To see C is nonzero, we remark that $[X,Y]=D$, so if $X^\#$ were 0, so would $D^\#$. □

Lemma 2.12. *Suppose $\mathfrak{b}_\alpha=\langle D,X\rangle<\mathfrak{j}_\alpha$, and that $\mathfrak{b}_\alpha$ acts on $\mathbb{F}^2$ with $X^\#=Cy\frac{\partial}{\partial x}$, C a nonzero constant. Let $D^\#=P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}$. Then $Q=Q(y)$ is a function of y alone, $Q(0)=0$, and $P(x,0)=(Q'(0)+\alpha)x+C_1$ for some constant C_1.*

Proof. We obtain restrictions on $D^\#$ by bracketing with $X^\#$ and using $[D^\#,X^\#]=-\alpha X^\#$ (the sign change is due to actions being on the left). We obtain

$$\left(CQ-Cy\frac{\partial P}{\partial x}\right)\frac{\partial}{\partial x}-Cy\frac{\partial Q}{\partial x}\frac{\partial}{\partial y}=-\alpha Cy\frac{\partial}{\partial x}. \qquad (1)$$

From the $\frac{\partial}{\partial y}$ coefficient in equation (1), it follows that $\frac{\partial Q}{\partial x}=0$ and so $Q=Q(y)$ only. By continuity this is true even at $y=0$.

From the $\frac{\partial}{\partial x}$ coefficient in equation (1), we obtain that $Q(0)=0$. Also, for $y\neq 0$,

$$y\frac{\partial P}{\partial x}=Q(y)+\alpha y.$$

For each y, this is independent of x. In particular, taking the limit as $y\to 0$ by L'Hospital's rule, $\frac{\partial P}{\partial x}|_{y=0}=Q'(0)+\alpha$, giving the desired result. □

Lemma 2.13. *Suppose $\mathfrak{j}_\alpha$ acts on $\mathbb{F}^2$ and Lemma 2.12 applies to $\mathfrak{b}_\alpha=\langle D,X\rangle$. Let $Y^\#=R(x,y)\frac{\partial}{\partial x}+S(x,y)\frac{\partial}{\partial y}$. Then in fact $Q'(0)=-\alpha/2$ and $P(x,0)=CS(x,0)=(\alpha/2)x+C_1$ for some constant C_1.*

Proof. We proceed as in the proof of Lemma 2.12, but now also using the relation $[Y^\#,X^\#]=D^\#$ (since $[X,Y]=D$). Equating $\frac{\partial}{\partial y}$ coordinates yields

$$-Cy\frac{\partial S}{\partial x}=Q$$

from which we conclude (again via L'Hospital's rule) that

$$C\frac{\partial S}{\partial x}|_{y=0} = -Q'(0).$$

Equating $\frac{\partial}{\partial x}$ coordinates yields

$$C(S - y\frac{\partial R}{\partial x}) = P,$$

so $P(x,0) = CS(x,0)$. Thus by Lemma 2.12,

$$Q'(0) + \alpha = \frac{\partial P}{\partial x}|_{y=0} = C\frac{\partial S}{\partial x}|_{y=0} = -Q'(0)$$

implying $Q'(0) = -\alpha/2$, and the result follows. □

Part 1 of Theorem 2.9 follows directly from the above Lemmas. The required rotation and translation is that given by the Lemma 2.11 to ensure Am lies on the x-axis, composed by a translation along the x-axis to eliminate the constant C_1 in Lemma 2.12.

Lemma 2.14. *Suppose J_α acts on $\mathbb{F}^{k+1}$ so that the N action in linear. The kernel V of the $\mathfrak{n}$ action is an A-invariant affine subspace. Up to a suitable translation, V is a linear subspace on which $\mathfrak{a}$ acts linearly.*

Proof. We essentially repeat the above arguments, with more complicated notation to allow for higher dimension.

By Lemma 2.10, V is A-invariant. The linearity of the N-action implies that we can without loss of generality consider coordinates

$$(x_1,\ldots,x_l,y_1,\ldots,y_m) = (x,y)$$

for $\mathbb{F}^{k+1}$ such that V is the hyperplane $y = 0$. The argument of Lemma 2.11 applies to show that we may take $X^\# = L(y)\cdot\frac{\partial}{\partial x}$, where $L(y)$ is a linear (matrix) function of y and $\frac{\partial}{\partial x}$ represents the vector of $\frac{\partial}{\partial x_i}$. (If $X^\#$ has a rotational component about V, we can ignore it since it leaves V invariant.)

Now we repeat the proof of Lemma 2.12 with our "multi-"coordinates x and y. We obtain the following matrix equations, with P and Q l- and m-vectors of functions of x and y such that $D^\# = P\cdot\partial/\partial x + Q\cdot\partial/\partial y$.

$$\frac{\partial P}{\partial x}L = \frac{\partial L}{\partial y}Q + \alpha L$$

and

$$\frac{\partial Q}{\partial x} L = 0.$$

Since L is of full rank (otherwise V would be bigger), the second equation implies $Q = Q(y)$ is independent of x. Since $\partial L/\partial y$ is a constant matrix, the right hand side of the first equation is independent of x. Thus $\partial P/\partial x$ is also independent of x, and $\mathbf{P} = \partial P/\partial x|_{y=0}$ is a constant matrix. Finally, $D^{\#}|V = (\mathbf{P}x + C_1) \cdot \partial/\partial x$, where C_1 is a vector of constants which can be eliminated by a suitable translation of V. □

Lemma 2.15. *The matrix* $\mathbf{P}$ *is diagonalizable over* $\mathbb{C}$ *and hence* V *is the direct sum of* A*-invariant lines.*

Proof. This is not completely automatic, since Jordan decomposition requires a representation of a full semisimple group and we only have one of AN on V. However, it follows easily by harnessing $Y^{\#}$ in the spirit of Lemma 2.13.

We need to show that that the matrix $\mathbf{P}$ of $D^{\#}|V$ has no generalized (i.e., higher-order) eigenspaces. It suffices to reach a contradiction in the case where $x = (x_1, x_2)$ and

$$\mathbf{P} = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}.$$

We thus take

$$\begin{aligned} D^{\#} &= P_1 \, \partial/\partial x_1 + P_2 \, \partial/\partial x_2 + Q \cdot \partial/\partial y, \\ X^{\#} &= L_1(y) \, \partial/\partial x_1 + L_2(y) \, \partial/\partial x_2, \\ Y^{\#} &= R_1 \, \partial/\partial x_1 + R_2 \, \partial/\partial x_2 + S \cdot \partial/\partial y. \end{aligned}$$

As in the prior Lemma, L_1 and L_2 are linear functions of y. Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be their derivatives; we may assume they are nonzero. As in Lemma 2.13, we must have $[Y^{\#}, X^{\#}] = D^{\#}$. Equating $\partial/\partial x_1$ and $\partial/\partial x_2$ coefficients we obtain

$$\mathbf{L}_1 S - L_1 \frac{\partial R_1}{\partial x_1} - L_2 \frac{\partial R_1}{\partial x_2} = P_1, \tag{2}$$

$$\mathbf{L}_2 S - L_1 \frac{\partial R_2}{\partial x_1} - L_2 \frac{\partial R_2}{\partial x_2} = P_2. \tag{3}$$

Now set $y = 0$; we have $P_1 = \lambda x_1 + x_2$ and $P_2 = \lambda x$. But all but the first terms in the left-hand sides of the above equations vanish, and in one case we obtain that S depends on x_2 and in the other that it does not, a contradiction. □

Lemma 2.16 (Analog of Corollary 2.8). *Suppose* J_α *acts on* $\mathbb{F}^{k+1}$*, so that the* N *action is linear. Suppose further that* W *is a* A*-invariant line of* N*-fixed points on which* A *acts linearly. Let* λ *be the eigenvalue of the* D*-action on* W*. Then* $\lambda \geq 0$*. Furthermore* $\lambda = 0$ *iff* W *is pointwise fixed by all of* J_α*.*

Proof. Let $m \in W$. If m is fixed by AN then the result follows from Proposition 2.5. If m is stabilized by precisely by N (or by N and a finite subgroup of A), then we can apply part 1 of Theorem 2.9 after choosing canonical local coordinates (x,y) on $J_\alpha m = \mathbb{F}^2 - \{0\}$ in which the N action is linear. In these coordinates, $D^\# = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ and $D^\#|W = (\alpha/2)x\frac{\partial}{\partial x}$.

Without loss of generality, suppose the ambient space $\mathbb{F}^{k+1}$ has coordinates $(e_0,\dots,e_k)$ with W is being the e_0-axis. The N action, given by $X^\#$, is linear in these coordinates as well. We have $D^\# e_0 = P\frac{\partial e_0}{\partial x} + Q\frac{\partial e_0}{\partial y}$. Letting $y = 0$ and $e_0 = e_0(x,y)$ we get

$$(\alpha/2)x\frac{\partial e_0}{\partial x}\Big|_{y=0} = \lambda e_0(x,0).$$

This differential equation implies that $e_0(x,0) = Cx^{2\lambda/\alpha} = Cx^p$ for some power p. By a scaling transformation, we can assume that $C = 1$ and so $e_0 = x^p + O(y)$.

We now show that p is a positive integer, implying $\lambda = p\alpha/2 > 0$. We have $X^\# = y\frac{\partial}{\partial x}$, so

$$X^\# e_0 = ypx^{p-1} + O(y^2).$$

(Note that $X^\# e_0$ is the $\partial/\partial e_0$ component of $X^\#$; this is quite different from $X^\#$ restricted to the e_0-axis, which by construction is 0.) Up to a linear transformation of coordinates of the e_i's, we may suppose that in e-coordinates, $X^\# e_0 = e_1$, i.e. that $e_1 = yx^{p-1}$. This includes a scaling to get rid of any multiplicative constant. Repeating the argument to compute successively $X^\# e_1, X^\# e_2, \dots$, we obtain that

$$e_i = p(p-1)\dots(p-i+1)y^i x^{p-i} + O(y^{i+1}).$$

(again up to a linear transformation of coordinates), up to such i for which the right hand side of this equation is 0. Since there is a last e_i, namely e_{k-1}, we must obtain such a zero eventually, and so p must be a positive integer $\leq k$. □

In particular, this Lemma shows the eigenvalues are nonnegative real numbers, and hence $D^\#|V$ is diagonalizable over $\mathbb{R}$ and part 2 of Theorem 2.9 follows.

To prove part 3, it merely remains to check the above arguments apply to local actions which are not *a priori* valid outside some neighbourhood of an open region containing an N-fixed point. The argument of Lemmas 2.12 through 2.14 carries over without difficulty once Lemma 2.11 is established. Once this is done, we know $D^\#|V$ is linear on the open region and hence the local action on V can be extended to a global one, and the rest of the argument makes sense. The only issue in Lemma 2.11 is making sure the map ψ does not exit the region of definition. We can ensure this by replacing $X^\#$ by $-X^\#$ in the definition of ψ to ensure contraction, if necessary.

It is not clear exactly how far a $\mathfrak{n}$-linear $\mathfrak{j}_\alpha$-action may stray from a $\mathfrak{j}_\alpha$-representation away from $V = \ker X^\#$. Clearly some "local shearing" perturbations can take place while maintaining the linearity determined by the $\mathfrak{n}$-"slices". However, the bracket relations on $X^\#, Y^\#$, and $D^\#$ impose restrictions of some sort (via partial derivatives on the coefficient functions) on the $\mathfrak{j}_\alpha$ action everywhere.

Also, we have not made use of the relation $[D^\#, Y^\#] = \alpha Y^\#$ in the proof of Theorem 2.9, since it yields no new information on $D^\#|V$. It does impose conditions on $R(x,0) = Y^\# x|V$; the conditions it imposes on $S(x,0) = Y^\# y|V$ are already known from $[Y^\#, X^\#] = D^\#$.

3 Actions of larger groups

3.1 Parabolic subgroups

For completeness, we start by showing that parabolic subgroups of semisimple groups are c-epimorphic. In this case we do not need any structure theory of epimorphic subgroups, nor the analysis of N-fixed points of SL_2 actions we have developed.

Corollary 3.1. *Suppose that G is a semisimple Lie group (with no compact factors) acting on M, that $m \in M$ is fixed by a parabolic P, and that the P-action locally linearizes near m. Then m is a G-fixed point.*

Proof. Let $U = \mathcal{R}_u P$ and $U^- = \mathcal{R}_u P^-$, where P^- is the parabolic opposite P. Suppose m is not fixed by G. Since $\langle P, U^- \rangle = G$, we can find a $g \in U_\alpha$ ($\alpha < 0$) which does not fix m. Let $J = \langle U_\alpha, U_{-\alpha} \rangle$ be the copy of SL_2 which intersects P in B, and in which g is lower-triangular. By Proposition 2.5 applied to J, g fixes m, a contradiction. □

By part 1 of Proposition 1.3, the same argument applies to epimorphic subgroups of semisimple groups containing a maximal torus.

3.2 T-normalized epimorphic subgroups

Let us use the notation of Proposition 1.3. The remaining issue is to show that if m is fixed by S, it is fixed by all of T. We need one trivial lemma:

Lemma 3.2. *Suppose D_1 and D_2 are commuting elements of a Lie algebra acting on $\mathbb{R}$, with $D_1^\# = (C_1 x + C_3)\frac{\partial}{\partial x}$ and $D_2^\# = C_2 x \frac{\partial}{\partial x}$, where C_1, C_2, C_3 are constants. Then $C_3 = 0$.*

Proof. This follows immediately from computing $[D_1^\#, D_2^\#]$, which must be 0 since $[D_1, D_2] = 0$. □

Theorem 3.3. *Suppose G is semisimple over* $\mathbb{R}$ *with no compact factors, and H is an epimorphic subgroup normalized by a maximal torus T of G. If G acts twice-differentiably on M, H fixes a point* $m \in M$*, and the H-action locally linearizes in a neighbourhood of m, then m is fixed by all of G.*

Proof. The proof is a direct analog of the proof in [BB3] of the converse of Proposition 1.3 in this paper, except instead of applying Corollary 2.8 to the whole space of a linear transformation to show m is fixed by T, we instead apply Theorem 2.9 to the kernel of the H action.

In view of Proposition 1.3 and Corollary 1.4, we assume without loss of generality that $H = SU$ and $G = \langle S, U, U^- \rangle$, with $U = U_\Psi$.

Let V be the kernel of the U action (the common kernel of the U_α actions), which contains m. Since T normalizes U, by Lemma 2.10, V is a T-invariant set. Furthermore, there are local coordinates ϕ valid in a neighbourhood of m in which $\phi(V)$ is a linear subspace. We henceforth omit ϕ.

Let $V' \subset V$ be the kernel of the H action. We now apply Theorem 2.9 to each $\mathfrak{j}_\alpha = \langle \mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha} \rangle$ for $\alpha \in \Psi$. Let $\mathfrak{t}'$ be the maximal torus of the subalgebra generated by all the $\mathfrak{j}_\alpha$'s, as in Corollary 1.4. We conclude that V' is an affine subspace on which the T' action is affine-linear. By definition V' is fixed by S. Thus in fact the T action on V' is linear after a suitable translation of coordinates in ϕ; to see this we merely need to verify that we can simultaneously translate to remove the constants C_1 at the end of the proof of Lemma 2.14, over all $\alpha \in \Psi$. This follows from Lemma 3.2.

Since T is commutative, we conclude further by Theorem 2.9 that V' is a direct sum of eigenspaces of $\mathfrak{t}$ with real eigenvalues in $X(T)$. Suppose now that W is a T-invariant line in V' and let $\lambda \in X(T)$ be the eigenvalue. Part 2 of Theorem 2.9 implies that $\lambda(D_\alpha)$ and hence $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Psi$. Since W is fixed pointwise by S, $\lambda | S = 0$ and hence by Proposition 1.3, $\lambda = 0$ and so W is fixed by T.

Thus m is fixed by U^- by Proposition 2.5, and so by G. □

3.3 *ng*-epimorphic subgroups

In the preceding argument, T-normalization is used in two ways.

1. To latch into a structure theory for the solvable epimorphic H.

2. To show that the T' action on V' is (locally) linear.

We now discuss how much further we can push the argument while weakening this hypothesis, as well as the hypothesis that G is semisimple with no compact factors.

Definition 3.4. *Suppose G is a real algebraic group and $H < G$. Call H ng-*epimorphic *(for* normalized-generated*) if $H <_e G$, and there exist semisimple subgroups $S_1, \ldots, S_n < G$ with no compact factors such that*

1. *$H_i = H \cap S_i <_e S_i$ for each i;*
2. *H_i is normalized by a maximal torus of S_i for each i; and*
3. *$\langle S_1, \ldots, S_n, H \rangle = G$.*

The following are immediate consequences of Theorem 3.3 and of the argument at the end of Section 1.7:

Corollary 3.5. *Suppose G is a real algebraic group, H is an algebraic subgroup, and H is ng-epimorphic in G. Then H is c-epimorphic.*

Corollary 3.6. *Suppose G is a real algebraic group and $H < G$ is an algebraic subgroup. Suppose H is c-observable (in G) and let L be its observable hull. If H is ng-epimorphic in L, then H is observable in G (and $H = L$).*

3.4 Example of a *ng*-epimorphic subgroup

Example 3.7. *([Moz]) Consider the 3-dimensional subgroup $H < SL_n$ with Lie algebra generated by*

$$X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ & \vdots & \end{pmatrix}, \text{ and}$$

$$D = \begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & 1-n \end{pmatrix}$$

D and X are the canonical basis for the image of the Borel subalgebra $\mathfrak{b} < \mathfrak{sl}_2$ in the standard irreducible representation of SL_2 on $\mathbb{R}^n$. Label this copy of $SL_2 < SL_n$ as J. Let $N' = \exp tZ$. Since the representation giving J is irreducible,

$\langle J, N' \rangle = SL_n$. The constructed subgroup is ng-epimorphic since it intersects J in a Borel subgroup and includes N'. It is, however, not T-normalized.

In SL_3 (c.f. Example 0.4 (3)), D, U, and U^- also generate a copy J' of $SL_2 < SL_3$ with $\langle J, J' \rangle = SL_3$ but this is no longer true in higher rank.

Larger *ng*-epimorphic but non-T-normalized examples can be generated along the same lines by replacing J by higher-rank semisimple subgroups G' and $\langle \exp sD, \exp tX \rangle$ by an epimorphic subgroup of G' normalized by a maximal torus of G' (but not necessarily a parabolic).

A similar construction to this example is used in [BB1, BB3] to show that any semisimple group contains a 3-dimensional epimorphic subgroup and an analogous argument shows this subgroup is *c*-epimorphic.

Corollary 3.8. *Any semisimple G with no compact factors has a three-dimensional minimal algebraic c-epimorphic subgroup.*

If G is not SL_2, there are no 2-dimensional epimorphic subgroups.

3.5 *ng*-epimorphic subgroups of nonsemisimple G

Proposition 3.9. *Let G be a real algebraic group, and write $G = ZCSN$, where $N = \mathcal{R}_u(G)$, ZCS is a (reductive) Levi component, $Z = \mathcal{R}(ZCS)$ the connected center, C is compact semisimple, and S is semisimple with no compact factors. Suppose $H <_e G$. Let $H_1 = H \cap SN$. Then*

1. *$H_1 <_e SN$;*
2. *$\langle SN, H \rangle = G$; and*
3. *if $H_1 < SN$ is ng-epimorphic, then so is $H < G$.*

To prove this, we use two epimorphicity reduction results of Barak Weiss.

Proposition 3.10 ([Wei, Theorem 5]). *Suppose $H < G$ are any algebraic groups. Let G_0 be the subgroup of G generated by unipotent elements, and $H_0 = H \cap G_0$. Then $H <_e G$ iff $H_0 <_e G_0$ and $G = HG_0$.*

Proposition 3.11 ([Wei, Theorem 4]). *Suppose $H < G$ are algebraic groups. Let $G_0 = \bigcap_{\chi \in X(G)} \ker(\chi)$ and $H_0 = H \cap G_0$. Then $H <_e G$ iff $H_0 <_e G_0$ and $G = HG_0$.*

Proof of Proposition 3.9. Let G_0 be the subgroup of G generated by unipotent elements, and

$$G_1 = \bigcap_{\chi \in X(G_0)} \ker(\chi) = SN.$$

Let $H_1 = H \cap G_1$. By the above two results, $H <_e G$ iff $H_1 <_e G_1$ and $\langle G_1, H \rangle = G$. If $H_1 < G_1$ is *ng*-epimorphic with $S_1, \ldots, S_n$ the requisite semisimple subgroups, then $\langle S_1, \ldots, S_n, H \rangle = \langle G_1, H \rangle = G$ and $S_1, \ldots, S_n$ apply to show $H < G$ is *ng*-epimorphic. □

By Proposition 1.1, we may suppose H_1 is $\mathbb{R}$-split solvable.

By Corollaries 3.5, 3.6, and Proposition 1.7, the following two Conjectures together would prove Conjecture 0.1 in full generality.

Conjecture 3.12. *Suppose $H < G$, $G = SN$ in the notation of Proposition 3.9. Then $H <_e G$ iff $H \cap S <_e S$ and $G = SH$, up to possibly replacing S by some conjugate.*

Conjecture 3.13. *Suppose G is semisimple with no compact factors. Then $H < G$ is epimorphic iff it is ng-epimorphic.*

These conjectures are true for all examples of epimorphic subgroups known to the author. The remainder of the paper consists of proving the conjectures in the cases covered by the hypotheses of Theorems 0.2 and 0.3. In the "G small" case, this consists of computations in low rank which are the content of the Section 4 and Appendix A. For the "H big" case, this is shown immediately below.

We remark that Bien and Borel have made various representation-theoretic conjectures on epimorphic subgroups, related to finite-dimensionality of induced representations (see [BB3] and [BB2]). The above two conjectures are more combinatorial in nature, and do not obviously follow from theirs.

3.6 "H big" implies *ng*-epimorphic

If G is reductive, then $N = 0$ in the decomposition of Proposition 3.9. Thus if $H <_e G$ is normalized by a maximal split torus of G, $H \cap S <_e S$ and is normalized by a maximal split torus of S and hence is c-epimorphic. This proves Theorem 0.2 in the "H big" case.

If H is c-observable and contains a maximal split torus of G, then by [Gro, Lemma 3.10] the observable hull L of H is reductive. Thus H is *ng*-epimorphic in L and hence observable in G. This proves Theorem 0.3 in the "H big" case.

In fact, the only obstruction to relaxing the condition "H contains T" to "H is normalized by T" in Theorem 0.3 is Conjecture 3.12, since by [Pom, Theorem 3.4], L is T-normalized whenever H is. In fact, in this case L can be determined precisely from information on which root spaces lie in H.

4 *ng*-epimorphic subgroups of low rank groups

4.1 Semisimple groups: rank 1 and reduction to simple

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{t} \oplus \mathfrak{n}$ be the Iwasawa decomposition, and $\mathfrak{h} \subset \mathfrak{t} \oplus \mathfrak{n}$ (since we may assume H is split-solvable without loss of generality). Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k$ and $\mathfrak{t} = \mathfrak{t}_1 \oplus \cdots \oplus \mathfrak{t}_k$ be the decompositions along the simple factors $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ of $\mathfrak{g}$. Denote $\mathfrak{s} = \mathfrak{h} \cap \mathfrak{t}$, $\mathfrak{u} = \mathfrak{h} \cap \mathfrak{n}$, $\mathfrak{u}_i = \mathfrak{h} \cap \mathfrak{n}_i$ and $\mathfrak{s}_i = \pi_i(\mathfrak{s})$, the projection onto $\mathfrak{g}_i$. The motivation for selecting out these components, in particular the projection for $\mathfrak{s}_i$, is apparent from the following example. With this notation, H is T-normalized iff $\mathfrak{h}$ is $\mathfrak{t}$-normalized.

Example 4.1 ([Wei]). $\mathfrak{h} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \times \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix} \right\} <_e \mathfrak{sl}_2 \times \mathfrak{sl}_2$. *In this example, the $\mathfrak{u}_i$ and the $\mathfrak{s}_i$ are the upper-triangular and diagonal subalgebras in each copy of $\mathfrak{sl}_2$.*

Lemma 4.2. *If $\mathfrak{h} <_e \mathfrak{g}$, then each of the $\mathfrak{u}_i$ and $\mathfrak{s}_i$ are nontrivial.*

Proof. Both $\mathfrak{t}$ and $\mathfrak{n}$ are nilpotent, hence observable, so if either of $\mathfrak{u}$ or $\mathfrak{s}$ were empty, $\mathfrak{h}$ would have a nontrivial observable hull. Since the projection of $\mathfrak{h}$ onto any $\mathfrak{g}_i$ is epimorphic there, the conclusion about the $\mathfrak{s}_i$ follows.

Now, $\mathfrak{t} + \mathfrak{h}$ is clearly epimorphic and $\mathfrak{t}$-normalized, and so it must include enough of $\mathfrak{n}$ so that together with the opposite roots it generates all of $\mathfrak{g}$. In particular, $\mathfrak{t} + \mathfrak{h}$ must nontrivially intersect each $\mathfrak{n}_i$. Since each $\mathfrak{n}_i$ is $\mathfrak{t}$-invariant, $\mathfrak{u}_i$ itself must be nonzero. □

Lemma 4.3. *Suppose each $\mathfrak{g}_i$ has the property that all of its epimorphic subalgebras are ng-epimorphic. Then $\mathfrak{g}$ has that property.*

Proof. Let $\mathfrak{g}_i' = \mathfrak{g}_i + \mathfrak{s}$, which is the direct sum of $\mathfrak{g}_i$ and a subtorus of $\mathfrak{t}$. Suppose $\mathfrak{h} <_e \mathfrak{g}$. Now $\mathfrak{h}_i' = \mathfrak{g}_i' \cap \mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}_i$ is certainly epimorphic in $\mathfrak{g}_i'$ and $\langle \mathfrak{h}_i' \rangle_i \subseteq \mathfrak{h}$. Since $\mathfrak{h}_i'$ includes all of $\mathfrak{g}_i'$ not in $\mathfrak{g}_i$, $\mathfrak{h}_i'$ is *ng*-epimorphic and hence so is $\mathfrak{h}$. □

Lemma 4.4. *If G is of rank 1, any $H <_e G$ contains T.*

Proof. $\mathfrak{t}$ is one-dimensional and hence $\mathfrak{s} = \mathfrak{t}$. □

4.2 The $\mathfrak{sl}_3$ case

Lemma 4.5. *If $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{sl}_3$, then $\mathfrak{h}$ is epimorphic iff it is ng-epimorphic.*

Proof. It suffices to prove that if $\mathfrak{h}$ is in the upper triangular Borel subalgebra and is not normalized by the maximal torus $\mathfrak{t}$, then it is as given in Example 3.7, up to the 1's in the basis vector X being replaced by other nonzero entries.

To prove this claim, we observe that the unipotent part $\mathfrak{u}$ of $\mathfrak{h}$ must have dimension 2, since if it had dimension 3 it would be normalized by $\mathfrak{t}$ and if it had dimension 0 or 1 the observable hull would clearly be smaller than $\mathfrak{sl}_3$. The diagonal part of $\mathfrak{h}$ must be a one-dimensional torus $\mathfrak{s}$ which normalizes $\mathfrak{u}$. Let $v \in \mathfrak{u}$ be an eigenvector for $\mathfrak{s}$. Since taking the Lie bracket with a $s \in \mathfrak{s}$ multiplies the component of v in root space by the the root evaluated on s, v has at most 2 nonzero components. Thus if $\mathfrak{u}$ is not normalized by $\mathfrak{t}$, $\mathfrak{u} = \mathfrak{g}_\alpha \oplus \mathbb{R}v$ for some root α and some vector v which intersects the other two positive root spaces.

Now $\mathfrak{u}$ must be closed under the Lie bracket, and this implies $\alpha = \alpha_{13}$, the highest root. Furthermore $\mathfrak{s} = \ker(\alpha_{12} - \alpha_{23})$ since it normalizes $\mathfrak{u}$. This proves the claim.

We reprove that $\mathfrak{h}$ is *ng*-epimorphic in the manner we shall generalize. There is a vector $v_- \in \mathfrak{g}_{21} \oplus \mathfrak{g}_{32}$ such that $\langle v, v_-, \mathfrak{s}\rangle$ is a copy of $\mathfrak{sl}_2$, as is $\langle \mathfrak{g}_{13}, \mathfrak{g}_{31}, \mathfrak{s}\rangle$. The intersection of $\mathfrak{h}$ with each of these copies of $\mathfrak{sl}_2$ is a Borel subalgebra and hence epimorphic.

Finally, $[v_-, \mathfrak{g}_{13}] \subset \mathfrak{g}_{12} \oplus \mathfrak{g}_{23}$ but is *not* spanned by v. Hence $\langle v, v_-, \mathfrak{g}_{13}\rangle \supseteq \mathfrak{g}_{12} \oplus \mathfrak{g}_{23}$ and hence is all of $\mathfrak{g}$. Thus $\mathfrak{h}$ is *ng*-epimorphic $\square$

4.3 Other rank 2 groups

The above argument generalizes to the other rank 2 simple groups in the hypotheses of Theorem 0.2. We outline the idea below; details, which culminate in a number of case-by-case computations, are in Appendix A.

A subspace $\mathfrak{u}$ of $\bigoplus \mathfrak{g}_\alpha$ ($\alpha \neq 0$) may directly contain certain of the root spaces $\mathfrak{g}_\alpha$. Call these α's contained roots of $\mathfrak{u}$. It will project nontrivially on certain other $\mathfrak{g}_\alpha$'s; call these α's its hybridized roots. For SL_3 as above, α_{13} is contained and α_{12} and α_{23} are hybridized. The subspace $\mathfrak{u}$ is normalized by $\mathfrak{t}$ iff there are no hybridized roots.

One can develop a combinatorial calculus using the geometry of the root space picture to find conditions on the hybridized and contained roots for the following to hold about $\mathfrak{u}$ and $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}$:

1. $\mathfrak{u}$ is normalized by $\mathfrak{s}$ (a perpendicularity condition);
2. $\mathfrak{h}$ is a subalgebra (arising from being closed under brackets);
3. $\mathfrak{h}$ is *ng*-epimorphic (generalizing the use of the vector v_- in the $\mathfrak{sl}_3$ case above); and

4. the smallest $\mathfrak{t}$-normalized subalgebra containing $\mathfrak{h}$ is epimorphic.

Then case-by-case analysis shows that for the other rank 2 simple algebras, any time the conditions of item 4 are satisfied (necessary if $\mathfrak{h} <_e \mathfrak{g}$), the conditions of item 3 are also satisfied and so $\mathfrak{h}$ is *ng*-epimorphic.

4.4 Quasiparabolics in SL_3

It remains to prove that epimorphic subgroups of observable subgroups of semsimple Lie groups with rank 1 and SL_3 factors are *ng*-epimorphic.

First suppose G is a product of rank 1 factors. Then the quasi-parabolics of G (Proposition 1.5) intersect any simple factor of G in either a reductive group or in a solvable one. Since solvable subgroups have no nontrivial epimorphic subgroups, the result is trivially true.

Now suppose $G = SL_3$. We investigate the possible quasiparabolics. $X(T)\big(= X(\mathfrak{t})\big) = \mathbb{Z} \oplus \mathbb{Z}$ and we may take any $\chi \in X(T)$ as a product (sum) of the linear functionals L_1, L_2, L_3 which pick out the diagonal entries. Let d_1 and d_2 be the differences between the coefficients of L_1 and L_2, and L_2 and L_3 respectively. By conjugation with the Weyl group, we may suppose $d_1, d_2 \geq 0$. Let $\mathfrak{q}_\chi$ be the Lie algebra of Q_χ. Then

$$\mathfrak{g}_{12} \oplus \mathfrak{g}_{23} \oplus \mathfrak{g}_{13} \subset \mathfrak{q}_\chi.$$

We have 3 cases:

1. $d_1 = d_2 = 0$. Then $\mathfrak{q}_\chi = \mathfrak{g}$. Observable subgroups L lying in Q_χ are reductive and we are done.

2. $d_1 > 0$ and $d_2 > 0$. Then no other root spaces lie in $\mathfrak{q}_\chi$, so Q and L are both solvable. L has no nontrivial epimorphic subgroups and we are done.

3. $d_2 = 0$ and $d_1 > 0$ (the opposite case is analogous). In this case Q_χ is the 3-dimensional analogue to Example 2.1. An observable $L < Q$ must have unipotent radical contained in $\mathfrak{g}_{12} \oplus \mathfrak{g}_{13}$.

   Now suppose $H <_e L$ in this case, $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}$, where $\mathfrak{s}$ is one-dimensional with 0 as the first entry. Since the values $\alpha(\mathfrak{s})$ with $\alpha = \alpha_{12}, \alpha_{13}, \alpha_{23}$ are different, $\mathfrak{u}$ is a direct sum of root spaces. If $\mathfrak{g}_{12} \oplus \mathfrak{g}_{23} \subset \mathfrak{u}$, then $\mathfrak{g}_{13} \subset \mathfrak{u}$ and the observable hull is not L. If $\mathfrak{g}_{12}$ is not in $\mathfrak{u}$ but $\mathfrak{g}_{23}$ is, then H is generated by $\mathfrak{g}_{23}$ and the standard copy of SL_2 in Q, which it intersects in a Borel subgroup B. We apply our epimorphicity result to B, and we are done. The remaining possibility is $\mathfrak{u} = \mathfrak{g}_{12} \oplus \mathfrak{g}_{13}$, which is subparabolic and hence observable, not epimorphic.

This concludes the proof of Theorem 0.3.

A Other rank 2 simple groups

Here we provide the details omitted from Section 4.3.

Let $\mathfrak{g}$ be a real rank 2 simple group, and suppose $\mathfrak{u}$ is a subspace of $\bigoplus \mathfrak{g}_\alpha$, the sum of the (nonzero) root spaces. For the moment, we suppose that each $\mathfrak{g}_\alpha$ is one-dimensional; we will remove this assumption later. We also suppose $\mathfrak{u}$ is normalized by a fixed one-dimensional torus $\mathfrak{s} \subseteq \mathfrak{t}$, i.e. $[\mathfrak{s}, \mathfrak{u}] \subseteq \mathfrak{u}$.

Call a specific α a *contained root* of $\mathfrak{u}$ if $\mathfrak{g}_\alpha \subseteq \mathfrak{u}$. Call $(\beta_1, \ldots, \beta_k) = (\beta_j)_{j=1,\ldots,k}$ *hybridized roots* of $\mathfrak{u}$ if there exists a subspace $V \subseteq \mathfrak{u}$ such that $V \subseteq \bigoplus \mathfrak{g}_{\beta_j}$, but this is not true if any of the β_j's are removed. The totality of all β_k arising (grouped in parentheses in any way) as hybridized roots are the β such that $\mathfrak{u}$ has a nontrivial projection onto β; we are using this apparently unwieldy definition to capture exactly which of the β are "indecomposably" grouped together. For notational convenience we sometimes write a contained root singly in parentheses and call it a "singly hybridized" root.

In this notation, the unique upper triangular non-$\mathfrak{t}$-normalized epimorphic subgroup of SL_3 has $(\alpha_{12}, \alpha_{23})$ as hybridized roots and α_{13} as a contained root.

Lemma A.1. *$\mathfrak{u}$ is normalized by $\mathfrak{s}$ iff the lines joining the endpoints of any 2 hybridized roots of $\mathfrak{u}$ are parallel, and perpendicular to $\mathfrak{s}$ when the root diagram of $\mathfrak{g}$ is embedded in $\mathfrak{t}$ via duality.*

Proof. Suppose $v \in \bigoplus \mathfrak{g}_\alpha$. Write $v = \sum v_\alpha x_\alpha$ where the x_α are some basis vectors for $\mathfrak{g}_\alpha$ and v_α the components of v with regards to this basis. Let $t \in \mathfrak{t}$. Then

$$[t, v] = \sum v_\alpha \alpha(t) x_\alpha. \tag{4}$$

Thus v is an eigenvector for $\mathfrak{s}$ iff $\alpha|\mathfrak{s}$ is the same for all α for which $v_\alpha \neq 0$. In particular, if $(\beta_j)_j$ are some hybridized roots and V the subspace of $\mathfrak{u}$ lying in their direct sum, then V is normalized by $\mathfrak{s}$ iff all $\beta_j|\mathfrak{s}$ are the same. This is true if all pairwise differences of the β_j are 0 on $\mathfrak{s}$, which is equivalent to the given perpendicularity condition. □

Lemma A.2. *$\mathfrak{u}$ is normalized by $\mathfrak{t}$ iff there are no hybridized roots (other than "singly hybridized", i.e. contained ones).*

Proof. By Lie bracketing $\mathfrak{u}$ with all of $\mathfrak{t}$, we obtain all the root spaces onto which $\mathfrak{u}$ has a nonzero projection. Thus $\mathfrak{u}$ is normalized by $\mathfrak{t}$ iff any hybridized root is contained. □

Lemma A.3. *Suppose $\mathfrak{w} \subseteq \mathfrak{u}$ is a one-dimensional subspace normalized by $\mathfrak{s}$, and that $\mathfrak{w}$ has hybridized roots $(\beta_j)_j$. Then there is a one-dimensional subspace*

$\mathfrak{w}_- \subseteq \mathfrak{g}_-$ which has hybridized roots $(-\beta_j)_j$, and such that $\langle \mathfrak{w}, \mathfrak{w}_- \rangle$ is a copy of $\mathfrak{sl}_2$ containing $\mathfrak{s}$, unless $\mathfrak{w}$ has a (single) contained root β and β and $\mathfrak{s}$ are perpendicular in the root diagram.

Proof. Since $[\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]$ lies in $\mathfrak{t}$, whatever are the components $\{v_\beta\}_\beta$ of the $\mathfrak{s}$-eigenvector $v \in \mathfrak{w}$, we can choose suitable coefficients for a vector $v_- \in \bigoplus -\mathfrak{g}_\beta$ such that $[v, v_-] \in \mathfrak{s}$, unless the Lie bracket of v with $\bigoplus -\mathfrak{g}_\beta$ lies in $\mathfrak{s}^\perp \subset \mathfrak{t}$, which can only occur if there is only one β and it is perpendicular to $\mathfrak{s}$. If this is not the case, then $\langle v, v_-, \mathfrak{s} \rangle$ is a copy of $\mathfrak{sl}_2$ as desired. □

Lemma A.4. *Suppose $\mathfrak{u}$ has hybridized roots $(\alpha_i)_i$ and $(\beta_j)_j$ (we allow contained, i.e. singly hybridized ones), then $[\mathfrak{u}, \mathfrak{u}]$ has as hybridized roots $\alpha_i + \beta_j$ (for any i, j for which this is nonzero), possibly partitioned into several parenthesized groups.*

Proof. This is just a consequence of the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ or 0, extended by linearity over vectors in hybridized root spaces. □

We will see that in the examples we need to compute, this partitioning into parenthesized groups can be determined ad-hoc. The following is now immediate, using Proposition 1.3 for item 3.

Proposition A.5.

1. *If $\mathfrak{u}$ is a $\mathfrak{s}$-normalized subspace, the smallest subalgebra containing $\mathfrak{u}$ is the closure of $\mathfrak{u}$ under the operation of Lemma A.4 (possibly with $\mathfrak{s}$ or $\mathfrak{t}$ as an additional summand).*
2. *If $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}$ is a $\mathfrak{s}$-normalized subalgebra, the smallest subalgebra $\mathfrak{h}'$ containing it which is normalized by $\mathfrak{t}$ is obtained by the following process:*
   - *(a) Replace any hybridized roots $(\beta_1, \ldots, \beta_k)$ with contained roots $\beta_1, \ldots, \beta_k$.*
   - *(b) Form the closure (as in item 1) to determine $\mathfrak{u}'$, and $\mathfrak{h}' = \mathfrak{s} \oplus \mathfrak{u}$.*
3. *If $\mathfrak{h}$ is also epimorphic, then so is $\mathfrak{h}'$ and hence the closure of $\mathfrak{u}' + \mathfrak{u}'_-$ is the whole root diagram.*
4. *Let $\mathfrak{g}'$ be the closure of $\mathfrak{h}$ together with the following:*
   - *(a) $(-\beta_j)_j$ as hybridized roots whenever $\mathfrak{u}$ has hybridized roots $(\beta_j)_j$; and*
   - *(b) $-\alpha$ as a contained root whenever $\mathfrak{u}$ has α as a contained root and α is not perpendicular to $\mathfrak{s}$.*

   If $\mathfrak{g}' = \mathfrak{g}$, then $\mathfrak{h}$ is ng-epimorphic.

We conclude with one observation about the "partitioning" in Lemma A.4 in the construction in item 4 in the Proposition.

Lemma A.6. *If* $(\beta_j)_j$ *are hybridized roots in* $\mathfrak{u}$ *and* α *is a contained root in* $\mathfrak{u}$ *such that* $\{-\beta_j+\alpha\} = \{\beta_j\}$ *(some permutation), then all the* β_j *are (individually) contained roots in* $\mathfrak{g}'$.

Proof. By Lemma A.4, the β_j are hybridized in $[(-\beta_j)_j, \mathfrak{g}_\alpha]$. However, in order for $[v, v_-] \in \mathfrak{s}$, the coefficients of v_- have to be different than those of v and so if $x \in \mathfrak{g}_\alpha$, $[v_-, x]$ is not in the span of v. Flipping back across $\mathfrak{t}$ and repeating this same calculation iteratively, we eventually get that a full-dimensional subspace of $\oplus \mathfrak{g}_\beta$ lies in $\mathfrak{g}'$ and hence the β_j must be contained roots. □

We now claim that for the rank 2 examples $SL_3(\mathbb{R})$, $Sp(4,\mathbb{R})$, $SO(p,2)$, $SU(p,2)$, $Sp(p,2)$, or G_2 ($p \geq 2$), we can verify via Proposition A.5 that any $\mathfrak{u}$ which is closed (item 1) is either *ng*-epimorphic, or $\mathfrak{u}'$ and hence $\mathfrak{u}$ are not epimorphic at all. It suffices to verify this for $\mathfrak{u}$ lying in a standard Borel subgroup, i.e. in one half of the root system diagram. Furthermore, $\mathfrak{u}$ must be at least 2-dimensional (to be epimorphic) and have some hybridized, non-contained roots (to be non-$\mathfrak{t}$-normalized).

We have already seen the case $\mathfrak{g} = \mathfrak{sl}_3$. In the language of this section, we need consider only two cases, described as lying in the root system diagram (A_2).

1. $\mathfrak{u}$ has two adjacent hybridized roots and a contained root adjacent to one of them. This is not closed since by Lemma A.4 the “middle” root should be contained, not hybridized.

2. $\mathfrak{u}$ has two hybridized roots both adjacent to a contained root lying between them. This is *ng*-epimorphic (and is Example 3.7).

Suppose $\mathfrak{g} = \mathfrak{s}p(4,\mathbb{R})$. The root system diagram is $C_2 (= B_2$ rotated) (terminology as in [Kna, p. 105 and 365]). Computation as above shows that in this case $\mathfrak{u}$ must contain or have as hybridized roots two adjacent roots in the root system diagram, and if they are both hybridized there must be a third hybridized or contained root. If there are exactly two hybridized roots, then it is easy to chase through the operations and find that if $\mathfrak{h}'$ is epimorphic, then $\mathfrak{h}$ is *ng*-epimorphic.

There is one possible case (up to rotation of the diagram) of 3 hybridized roots (containing a 2 dimensional subspace) of three roots with collinear endpoints. In this case, Lemma A.6 applies to show that the 2 roots perpendicular to the middle hybridized one are contained, and this can be chased through to show that $\mathfrak{h}$ is again *ng*-epimorphic.

Exactly the same argument now applies for $\mathfrak{g}$ of type G_2, except that the 3 roots with collinear endpoints of the diagram C_2 are replaced by 4 roots with collinear endpoints, and if these are included the sum of the two middle ones is one of the long roots in between. Clearly $\mathfrak{u}$ cannot consist only of long roots, and hence any epimorphic $\mathfrak{h}$ is also *ng*-epimorphic.

We now treat the remaining cases. As discussed in [Fer, Ch. 7], these can be treated in a unified manner as $\mathfrak{su}(p,2)_{\mathbb{F}}$ where $\mathbb{F}$ is one of $\mathbb{R},\mathbb{C},\mathbb{H}$. Let $d = \dim_{\mathbb{R}}(\mathbb{F})$. The (restricted) root system diagram is $(BC)_2$, the union of B_2 and C_2. Now, however, the root spaces of the short roots have dimension (over $\mathbb{R}$) $(p-2)d$ and the root spaces of the medium roots have dimension d. The root spaces of the long roots have dimension 1. Furthermore, since $\mathfrak{t}$ is real, we can separate out the short and medium roots spaces into d layers along the vector space structure of $\mathbb{F}$ over $\mathbb{R}$ in such a way that Lie brackets of vectors within one layer remain in that layer. (The long root spaces belong to all layers.)

We now have to modify our development in this section to account for the remaining multidimensional restricted root spaces. The crucial part is the equation (4), in which we now have several different basis vectors x_α in $\mathfrak{g}_\alpha$. However, the coefficient $\alpha(t)$ will certainly be the same for all of them and the Lie bracket will not mix across layers. This means that the rest of the development can be done separately in each layer and for each of the $(p-2)$ dimensions in the root space of a medium root within one layer. Since this is the case, it suffices to prove *ng*-epimorphicity of all epimorphics in the abstract diagram (BC_2), ignoring the additional dimensions.

In this diagram, in order to have $\mathfrak{u}'$ epimorphic, we clearly need at least some (hybridized or contained) short roots, since the sum of only long and medium roots is never short. As in the previous cases, for $\mathfrak{u}'$ to be epimorphic and for $\mathfrak{u}$ to be a subalgebra, at least two roots adjacent in the diagram must be hybridized or contained. Analogous to the $\mathfrak{sp}(4,\mathbb{R}) = B_2$ case, there are now two possibilities for 3 hybridized roots, one with one short and 2 medium roots, and the other with one medium and two long roots. Case by case analysis shows that if we have hybridized and contained roots which yield $\mathfrak{h}'$ epimorphic, then $\mathfrak{h}$ is *ng*-epimorphic as required.

We remark that in the $\mathfrak{sl}_3(\mathbb{R})$ and $\mathfrak{sp}(4,\mathbb{R})$ cases, though we have not taken the time to state it, we have effectively classified all the possible algebraic solvable epimorphic subgroups in the process of showing they are *ng*-epimorphic. This is not quite the situation in the higher-dimensional restricted root space case, since we have sidestepped the issue of exactly how much of each root space must be included to get an epimorphic subgroup, merely showing that if so, it is also enough to make it *ng*-epimorphic. It is expected that the other exceptional restricted rank 2 simple algebras (see [Kna, p. 365]) could be dealt with a similar manner, assuming a similar "layering" of the root spaces holds, such that the dimension of the root spaces decreases with increasing length of the root in the restricted root system diagram.

References

[BBHM] A. Bialynicki-Birula, G. Hochschild, and G. Mostow. Extensions of representations of algebraic linear groups. *Amer. Jour. Math.* **85** (1963), 131–144.

[BB1] F. Bien and A. Borel. Sous-groupes épimorphiques de groupes linéaires algébriques. I. *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992), 649–653.

[BB2] F. Bien and A. Borel. Sous-groupes épimorphiques de groupes linéaires algébriques. II. *C. R. Acad. Sci. Paris Sér. I Math.* **315** (1992), 1341–1346.

[BB3] F. Bien and A. Borel. Epimorphic subgroups of affine algebraic groups. preprint, 1995.

[Bor] A. Borel. *Linear algebraic groups*. Springer-Verlag, New York, second edition, 1991.

[Fer] R. Feres. *Dynamical Systems and Semisimple Groups: an introduction*. CUP, Cambridge, 1998.

[Gro] F. D. Grosshans. *Algebraic homogeneous spaces and invariant theory*. Springer-Verlag, Berlin, 1997.

[Kna] A. W. Knapp. *Lie groups beyond an introduction*. Birkhäuser, Boston, 1996.

[KN] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Interscience, New York, 1961 (vol. I) and 1969 (vol. II).

[Moz] S. Mozes. Epimorphic subgroups and invariant measures. *Ergodic Theory Dynam. Systems* **15** (1995), 1207–1210.

[Per] M. Pergler. Epimorphic subgroups of SL_n, trees, and the Tits boundary. preprint, 1999.

[Pom] K. Pommerening. Observable radizielle Untergruppen von halbeinfachen algebraischen Gruppen. *Math. Zeitschrift* **165** (1979), 243–250.

[SW] N. Shah and B. Weiss. On actions of epimorphic subgroups on homogenous spaces. preprint.

[Suk] A. A. Sukhanov. The description of observable subgroups of linear algebraic groups. *Mat. Sb. (N.S.)* **137(179)** (1988), 90–102, 144.

[Sza] J. P. Szaro. Isotropy of semisimple group actions on manifolds with geometric structure. *Amer. J. Math.* **120** (1998), 129–158.

[Wei] B. Weiss. Finite-dimensional representations and subgroup actions on homogeneous spaces. *Israel J. Math.* **106** (1998), 189–207.

[Zim1] R. J. Zimmer. Actions of semisimple groups and discrete subgroups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 1247–1258, Providence, RI, 1987. Amer. Math. Soc.

[Zim2] R. J. Zimmer. Ergodic theory, Groups, and Geometry. Lecture notes from NSF-CBMS Regional Conference in the Mathematical Sciences, University of Minnesota, June 1998. To appear in book form.