

EFFECT OF SMALL GAS MIXTURE CHANGES OF PITCH OF SOUND (PRELIMINARY)

MARTIN PERGLER

In a recent note [2], I provided a theoretical estimate of the effect on the pitch of a woodwind instrument caused by the replacement of O₂ by CO₂ in the lungs of the player over the course of a breath-phrase. I have since learned that this effect has been studied experimentally by Fuks [1], who also uses formulas after Nederveen to estimate its magnitude theoretically.

It appears that Nederveen's formulas are very similar to my own, though I have not seen them in their original form (or their derivation) at this time. However, their form in [1] may perhaps not make obvious certain features of the relationship which may or may not have as of yet been remarked on.

In this additional note, I reformulate the results of [2] from the more general standpoint of the effect of arbitrary small gas mixture changes on the pitch of sound. I am omitting a derivation of these generalizations in this preliminary version, pending whether this will be of any interest to anybody. However, the essential arguments are present in [2] and the only additional argument is some additional mathematics.

1. RESULTS

Briefly, small gas mixture changes result in a pitch change which is approximately linear in the magnitude of the mixture change. This change is independent of the base pitch and of temperature (provided the temperatures of the original and the resultant gas mixtures are the same). It is caused by changes in density and (adiabatic) compressibility of the gas mixture.

Suppose a gas mixture has components Gas₁, Gas₂, ..., Gas_n. A change in the mixture can be represented as a vector $\Delta p = (\Delta p_1, \dots, \Delta p_n)$ of differences in molar fractions of the components, such that $\sum_i \Delta p_i = 0$. Then the pitch change, provided all the Δp_i are "small", is

$$\text{Pitch}_{\Delta p} = \text{Pitch}_0 - 865 \sum_i \Delta p_i (X_i + Y_i) \text{ (in cents)}$$

(All quantities with subscript 0 refer to the value of the original gas mixture.) Here X_i is a "density adjustment factor" and Y_i is a "compressibility adjustment factor" (my terminology) for Gas_i relative to the original mixture.

The above set-up refers to (small) adjustments in the balance of components already present in the mixture. We can use the same formula to determine the effect of introducing small amounts of additional gases or vapours (in this case $\sum_i \Delta p_i > 0$), which reduce existing components proportionately.

2. DETERMINATION OF X_i AND Y_i

How do we find these adjustment factors from elementary theoretical or experimentally measurable (and commonly tabulated) properties of the gases? The "density adjustment factor" is given in

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terms of the densities of the gases or in terms of (average) molar weights:

$$X_i = \frac{D_i - D_0}{D_0} = \frac{m_i - m_0}{m_0}.$$

The “compressibility adjustment factor” is given in terms of specific heats or adiabatic constants:

$$Y_i = \frac{(C_V)_i}{(C_V)_0} - \frac{(C_P)_i}{(C_P)_0} = \left(\frac{\gamma_0 - 1}{\gamma_i - 1} \right) \left(1 - \frac{\gamma_i}{\gamma_0} \right).$$

It could also be expressed in terms of (“average”) degrees of freedom in the kinetic-molecular theory of gases, though this begins to deviate from experimental thermodynamic results for polyatomic gases. For diatomic gases relative to a mixture of diatomic gases, $Y_i = 0$.

Other relevant properties of the gas mixture can be expressed via these adjustment factors (density, compressibility, bulk modulus, sound velocity)

$$D_{\Delta p}/D_0 = \left(1 + \sum_i X_i \Delta p_i \right), \quad \text{comp}_{\Delta p}/\text{comp}_0 = \left(1 + \sum_i Y_i \Delta p_i \right),$$

$$B_{\Delta p}/B_0 = \gamma_{\Delta p}/\gamma_0 = \left(1 - \sum_i Y_i \Delta p_i \right), \quad v_{\Delta p}/v_0 = \left(1 - \frac{1}{2} \sum_i (X_i + Y_i) \Delta p_i \right).$$

The first two provide the rationale for my naming of X_i and Y_i .

3. VALUES FOR PULMONARY VERSUS TIDAL AIR

Suppose the initial mixture is atmospheric dry air, which is almost exclusively a mixture of 0.79 diatomic N_2 and 0.21 diatomic O_2 (refinements could be made for water vapour and the trace amounts of other gases present, particularly argon.) Then $X(O_2) = 0.11$, $X(CO_2) = 0.53$, $Y(O_2) = 0$ and $Y(CO_2) = 0.10$. (The value of Y for a monatomic gas is -0.12 . The value predicted by kinetic-molecular theory for a triatomic gas would be 0.12, slightly more than actually that of CO_2 .)

In [2], the situation was that of $\Delta p(CO_2) = p$ and $\Delta p(O_2) = -p$, so that ultimately $\text{Pitch}_{\Delta p} - \text{Pitch}_0 = -450p$. Not knowing of any other experimental results, I used $p = 0.05$ in keeping with normal (nonmusical) breathing. It appears [1] that for the deeper and longer breathing associated with playing a wind instrument, p increases to about this value in the course of the first few seconds of a breath, but ultimately rises up to about 0.085 in a very long breath-phrase.

4. NONLINEARITY

The calculations yielding the above equations use approximation by differentials and are thus only approximately valid, though the quality is very good if all $|\Delta p_i| \ll 1$. More precisely, the degree of the error in the approximation is given by the second-order Mean Value Theorem, or equivalently can be estimated by the second-order term in the Taylor expansion. For pitch, this term is $-432 \sum_i (X_i^2 \Delta p_i^2 + Y_i^2 \Delta p_i^2)$.

In the case in [2], this works to $-131p^2$ cents (almost all the nonlinearity due to X , not Y), which means the pitch effect is on the order of about “1 cent more than linear” at $p = 0.085$, less than 1/3 cent nonlinear at $p = 0.05$, but would be 5 cents nonlinear at $p = 0.2$.

A more significant imprecision in application of the above results, whether in the form of this note or of Nederveen, is expected to be in accounting for “missing molar fractions”. Suppose it is predicted that a change of the mixture Δp should occur, with $\sum_i \Delta p_i = 0$, and experimental measurements are taken to determine the Δp_i . For whatever reasons, the experimental values may have nonzero sum. Plugging in the experimental values into the formula is consistent with the interpretation that the “excess” or “missing” gas has the same composition as the original gas mixture, which may be rather dubious.

In my (currently incomplete) opinion, the formulas for these effects after Nederveen in [1] ultimately differ from those above in only a few aspects. The principal is in the incorporation of temperature differences, which can be taken care of separately via the Ideal Gas Law. The Nederveen formulas there are linearized around $0^{\circ}\text{C} = 273^{\circ}\text{K}$, while in many cases (e.g., woodwind exhalation effects) a base closer to room or body temperature would be more precise since the resultant nonlinearity (fortunately not propagated) is in excess of that due to the gas mixture approximations. Furthermore, this makes the calculation corresponding there to $(1 - \sum_i X_i \Delta p_i)$ here become needlessly complicated. Second, for reasons mentioned above, I prefer to determine Y_i from tabulated specific heat or adiabatic constant values for polyatomic gases rather than from degrees of freedom. Both of these differences, however, have minimal effect on the numerical results. Finally, the calculation can be continued to see the essentially linear effect on pitch, and to estimate the linearization error.

REFERENCES

- [1] Leonardo Fuks, "Prediction and measurement of exhaled air effects in the pitch of wind instruments", *Proceedings of the Institute of Acoustics*, 373-378 (1997), St Albans, United Kingdom.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO IL 60637
E-mail address: pergler@math.uchicago.edu